CONVERGENCES AND TOPOLOGIES FOR FAMILIES OF FUNCTIONS

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CHAPTER I

INTRODUCTION

Convergences for nets of functions have been fundamental tools in the study of function spaces. Undoubtedly, pointwise convergence, uniform convergence, and uniform convergence on compacta are the most familiar. In Chapter II we investigate these in addition to several other types of convergences found from time to time in the literature.

We attempt to add to the understanding and usefulness of these convergences by the exhibition of the relationships between the convergences. The multitude of implications is handled efficiently by means of convergence diagrams. A few such relationships appear in the literature usually for some of the convergences under conditions stronger than necessary for the results. Our conditions are minimal as is shown by the comprehensive list of counter-examples in section 5. Each implication shown on the diagrams is proven and each non implication can be verified by use of the tables of counter-examples.

Certain questions such as those concerning compactness of function spaces can be phrased in terms of convergences of nets of functions without placing a topology on the function space. Various "compactnesses" for function spaces have been a popular subject for research. Another alternative to topologizing the function space is to employ the convergence space structure as Poppe does in Poppe [18], and [19]. In such structures topological type properties can be defined.

Some of the convergences lead naturally to useful topologies on the function space. Among the most investigated are the topologies of uniform convergence, pointwise convergence, uniform convergence on compacta, and the compact open topology. Brace [7] has shown the usefulness of the topology of almost uniform convergence.

The space of all continuous real valued functions on a topological space X has been thoroughly investigated under the topologies mentioned above, with the exception of the topology of almost uniform convergence. In this thesis we do not restrict the function spaces to be a subset of the continuous functions. The continuity assumption is a strong one from the point of view of applications to analysis.

In Chapter III various topologies for function spaces are defined and relationships between the topologies are investigated. These relationships along with some other results are a direct application of the convergence diagrams. An Arzela type theorem is derived from the inspection of the convergence diagrams.

Conditions are found in Chapter IV which guarantee that a function space has certain properties such as metrizability, T_1, T_2, and separability, when the topologies of uniform convergence, pointwise convergence, and the compact open topology are placed on the function space. A general approach is developed and applied to the different topologies.

The notation conforms to that of Kelley [15] as much as is practical. Throughout this thesis X and Y are topological spaces with F(X,Y) denoting the set of all functions from X into Y and with C(X,Y) denoting the set of all continuous functions from X into Y. Continuity of a function or map is not be assumed unless specifically stated. The symbol "R" denotes the real numbers with the usual topology and metric and "R^n" denotes the n dimensional real vector space with the usual metric topology.

CHAPTER II

RELATIONS BETWEEN THE CONVERGENCES

Many types of convergences have interested the analyst. The natural settings vary from sequences of real valued functions to nets of functions from an abstract topological space to another. We have restricted this investigation to eleven types of convergences of nets of functions from a topological space into a uniform topological space. Poppe [18] defines these convergences in terms of filters.

1. Definitions of the Convergences

Let X be a topological space, $(Y,\mathcal{V})$ be a uniform topological space, $\{f_a\}$ be a net of functions from X into Y, and let f be a function from X into Y. The letters in parenthesis denote our abbreviation for the type of convergence.

(1.1) Definition: The net f_a converges pointwise (p) to f if and only if for each x in X the net $f_a(x)$ converges to $f(x)$.

(1.2) Definition: The net f_a converges strongly continuously (sc) to f if and only if $\{x_b\}$ is a net in X and $f(x_b)$ converges to y in Y then the net S defined by $S(a,b) = f_a(x_b)$, with the product direction, converges to y.

(1.3) Definition: The net f_a converges continuously (c) to f if and only if for each x in X and for each neighborhood V of $f(x)$ there is a neighborhood U of x and there is an index n so that if $a \geq n$ then $f_a(U) \subset V$.

(1.4) Definition: The net f_a converges uniformly (u) to f if and only if for each V in $\mathcal{V}$ there is an index n so that if $a \geq n$ then $(f_a(x), f(x))$ is in V for each x in X.

(1.5) Definition: The net f_a converges quasi-uniformly (qu) to f if and only if the convergence is pointwise and for each V in $\mathcal{V}$ and for each index m there are indices $a_1, \ldots, a_n$ each of which follow m such that for each x in X there is at least one index i between 1 and n so that $(f_{a_i}(x), f(x))$ is in V.

(1.6) Definition: The net f_a converges almost uniformly (au) to f if and only if each subnet converges quasi-uniformly to f.

(1.7) Definition: The net f_a converges locally uniformly (lu) to f if and only if for each x in X there is a neighborhood U of x so that the net converges uniformly to f on U.

(1.8) Definition: The net f_a converges pointwise uniformly (pu) to f if and only if for each x in X and for each V in $\mathcal{V}$ there is a neighborhood U of x and an index m such that $(f_a(y), f(y))$ is in V whenever $a \geq m$ and y is in U.

(1.9) Definition: The net f_a converges simply uniformly (su) to f if and only if the convergence is pointwise and for each x in X and for each V in $\mathcal{V}$ and for each index m there is a neighborhood U of x and an index $a \geq m$ such that $(f_a(y), f(x))$ is in V whenever y is in U.

(1.10) Definition: The net f_a converges uniformly on compacta (k) to f if and only if the convergence is uniform on each compact subset of X.

(1.11) Definition: The convergence of the net f_a to f is compact open (co) if and only if for each compact subset K of X and for each open set U in Y for which $f(K) \subset U$ there is an index m such that if $a \geq m$ then $f_a(K) \subset U$.

(1.12) Definition: A family of functions F from a topological space X into a topological space Y is evenly continuous if and only if for each x in X, y in Y, and neighborhood V of y there are neighborhoods W of y and U of x such that if f is in F and $f(x)$ is in W then $F(U) \subset V$. Let S be a net of functions from X into Y with directed set D. We will say the net S is evenly continuous if and only if the family $\{S(a): a \in D\}$ is evenly continuous.

(1.13) Definition: A family of functions F from a uniform topological space $(x, \mathcal{U})$ into a uniform topological space $(Y, \mathcal{V})$ is said to be uniformly equicontinuous if and only if for each V in $\mathcal{V}$ there is a U in $\mathcal{U}$ such that $(f(x), f(y))$ is in V whenever (x,y) is in U and f is in F. We say the net S with directed set D is uniformly equicontinuous if and only if the family $\{S(a): a \in D\}$ is uniformly equicontinuous.

2. General Remarks

Pointwise convergence, uniform convergence, and uniform convergence on compacta need little comment. A convergence such as quasi-uniform convergence may not be as well known. Under certain conditions quasi-uniform convergence is necessary and sufficient for a net of continuous functions to converge to a continuous function as is stated in Arzela's Theorem (see Theorem (2.1)). In Bartle [4], and Brace [5], relationships of quasi-uniform convergence to the weak and weak* topologies are found. Sirvint [20] uses quasi-uniform convergence to investigate weak compactness. Quasi-uniform convergence is not usually topological since a net may converge quasi-uniformly without each subnet converging quasi-uniformly (see Example (5.8)). Almost uniform convergence is obtained by requiring all subnets to converge quasi-uniformly. We show that almost uniform convergence is a topological convergence in Chapter III. Fuller [13] uses pointwise uniform convergence, uniform convergence on compacta, and continuous convergence (his quite continuous convergence) in his study of functional convergence. Poppe [18] uses all of the convergences in this thesis in his characterization of various types of compactness in X through convergences of functions in $C(X)$. Brace [6] shows relationships between almost uniform convergence and repeated limits. The different types of convergences seem to stem from many varied roots in analysis. Poppe [18] asserts that the original forms of pointwise uniform, quasi-uniform, almost uniform and continuous convergence, in sequential form, are contained in Hahn's Theorie der Reellen Funktionen which was published in 1921.

Undoubtedly some of the interest in quasi-uniform convergence and almost uniform convergence stems from Arzela's Theorem (Arzela [2,3]). See also Brace [7, Theorem 2.2] and Poppe [18, Theorem 1.3]. A modern version of the theorem is as follows.

(2.1) Theorem: Let X be a topological space, $(Y, \mathcal{U})$ be a uniform topological space and $\{f_a\}$ be a net of continuous functions which converges (p) to f. If the convergence of f_a to the function f is quasi-uniform on a subset B of X then f is continuous on B. Conversely, if f is continuous on X then the convergence is quasi-uniform on compacta.

An equivalent version of Arzela's theorem seems to have been unnoticed which suggests the importance of almost uniform convergence. We state it as follows.

(2.2) Theorem: Let X be a topological space, $(Y, \mathcal{U})$ be a uniform topological space and let $\{f_a\}$ be a net of continuous functions from X into Y. Suppose the net converges (p) to f. If the convergence is quasi-uniform on a subset B of X then f is continuous on B. Conversely, if f is continuous on X then the convergence is almost uniform on compacta.

Proof: The first part is proven in Theorem (4.16). Conversely suppose f is continuous. Without loss of generality we may assume that X is compact. It follows from Theorem (4.17) that any subnet converges (su) to f and from Theorem (4.24) it follows that that subnet converges (qu) to f. Hence all subnets of $\{f_a\}$ converge (qu) to f and the convergence is therefore almost uniform.

3. The Convergence Diagrams

Let X be a topological space, $(Y, \mathcal{U})$ be a uniform topological space, $\{f_a$: a is in D$\}$ be a net of functions from X into Y, and let f be a function from X into Y. The relationships between the types of convergences of the net f_a to f are shown graphically in the figures which follow under various combinations of conditions on X, Y, f, and the family $\{f_a\}$. The conditions are listed under each figure. The absence of a condition means that no condition is to be assumed for that figure other than the general conditions listed above. The arrows in each figure are sufficient to yield all implications quickly. If an implication cannot be obtained between two convergences by the combination of arrows in a given chart then that implication does not generally hold. The proofs of the implications follow the figures, followed by tables of counter examples sufficient to verify all of the non-implications. Example: If X is compact Figure 3 shows that continuous convergence implies quasi-uniform convergence. That quasi-uniform convergence does not imply continuous convergence follows from item 8 in Table 3 together with Figure 3.

The diagrams seem to give us insight into what conditions are crucial from the point of view of the convergences. It seems that compactness or local compactness on X and continuity on the f_a and f seem to be strongest factors. It is no wonder that the convergences have been useful in characterizing compactness in X. The range Y seems to have the least influence on the convergences. The majority of the counter examples have X and Y as subsets of the real numbers with the usual topology. Thus, such conditions as second countability, metrizability, and separability on X and Y seem to have little influence on the relationships. It is surprising to note that adding the uniform requirement to Figure 21 produces no new relationships in Figure 24 even with the even continuity replaced by the stronger uniform equicontinuity.

<u>Note</u>: In Figures 1-24 all convergences imply pointwise convergence.

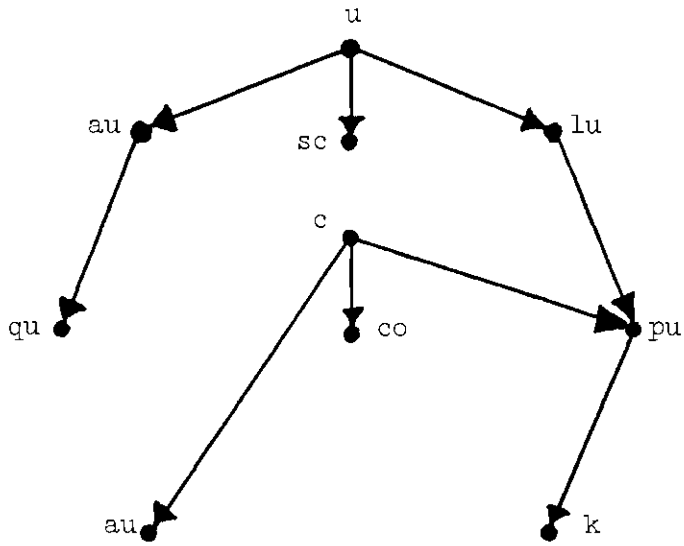

Figure 1. No Conditions

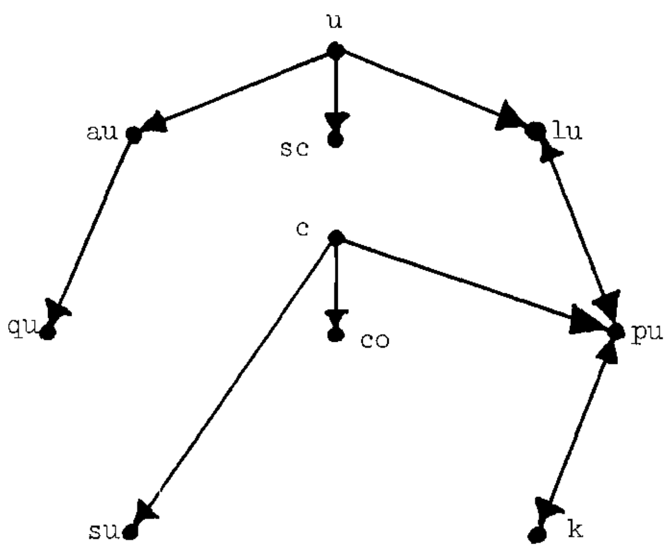

Figure 2. X Locally Compact

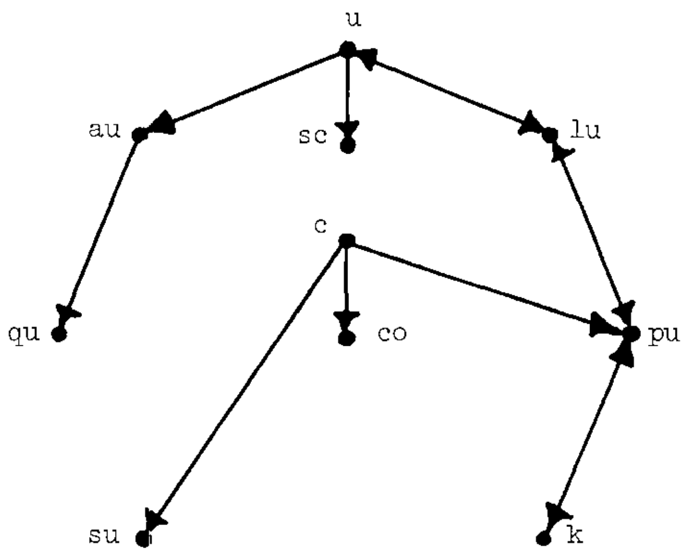

Figure 3. X Compact

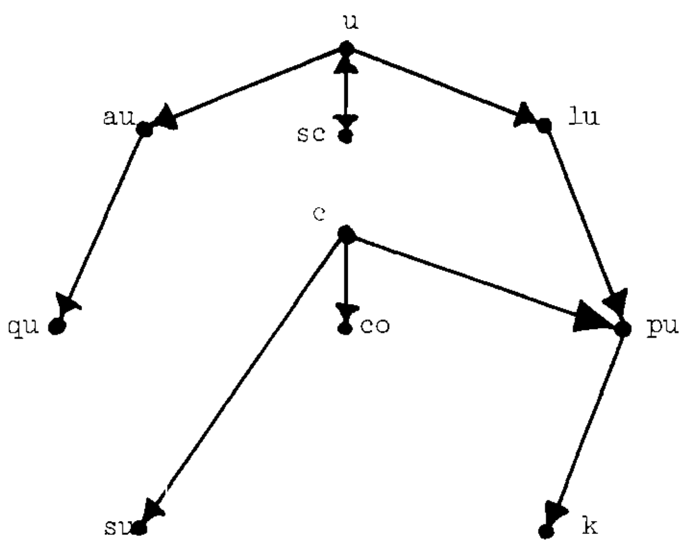

Figure 4. Y Compact

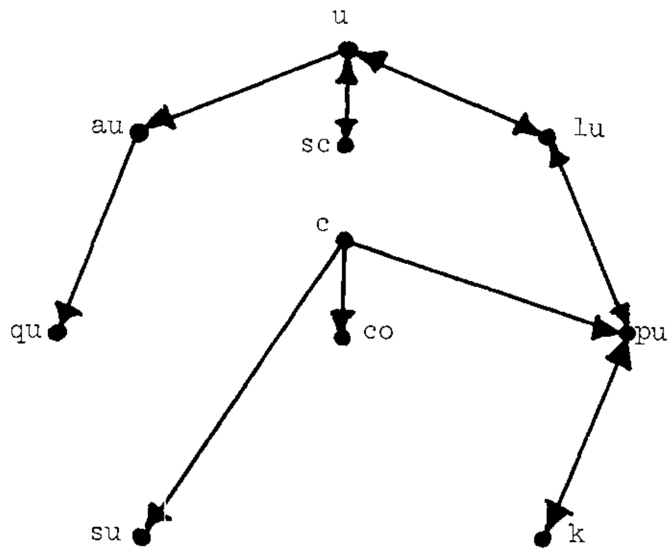

Figure 5. X and Y Compact

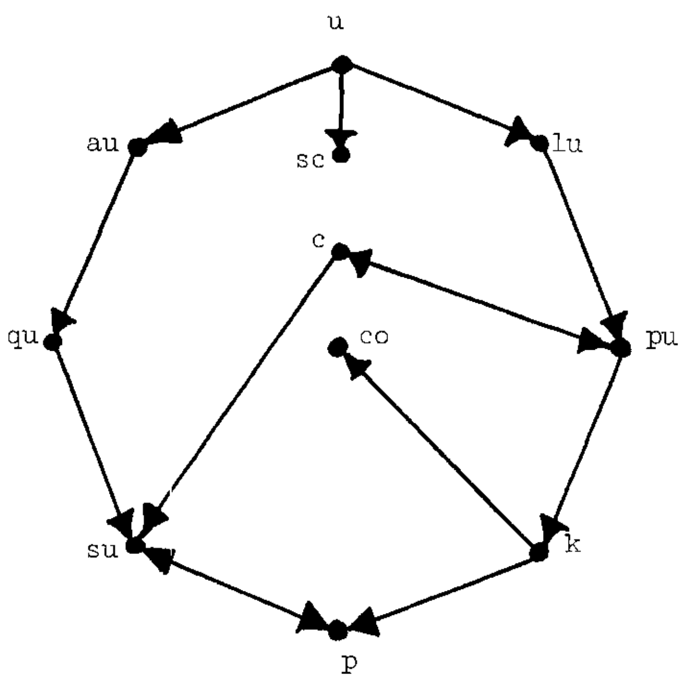

Figure 6. f_a Continuous

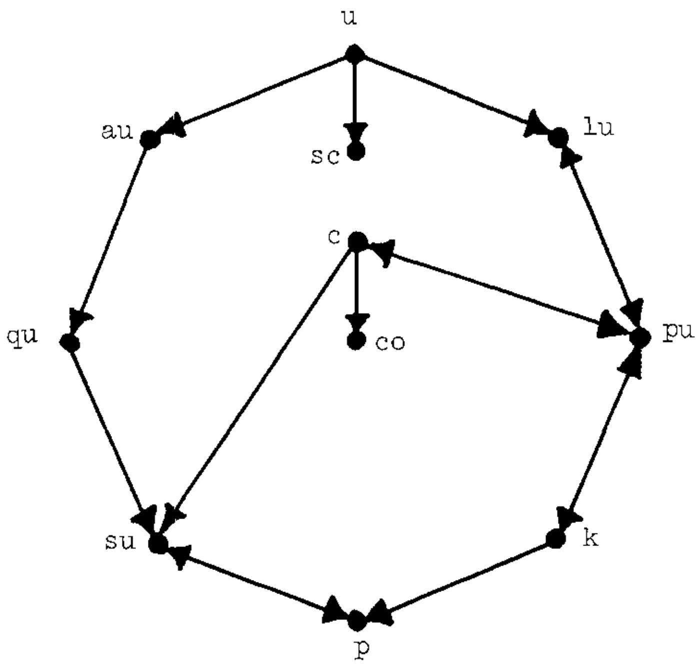

Figure 7. X Locally Compact, f_a Continuous

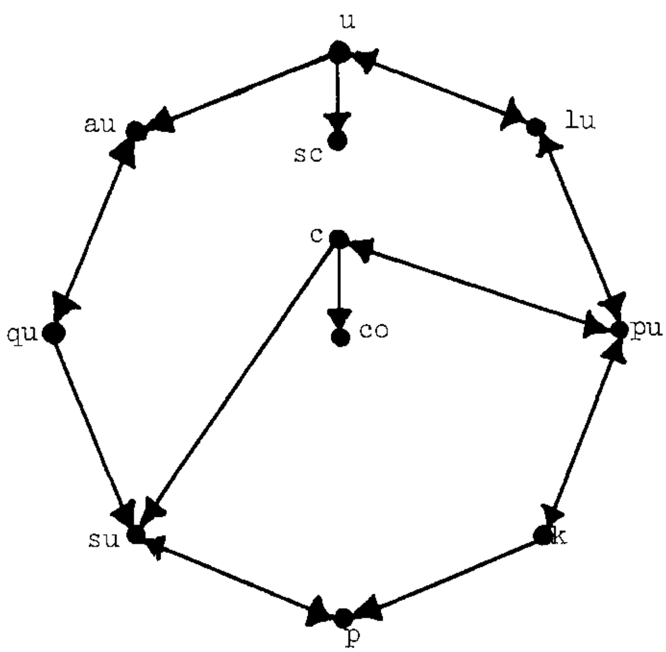

Figure 8. X Compact, f_a Continuous

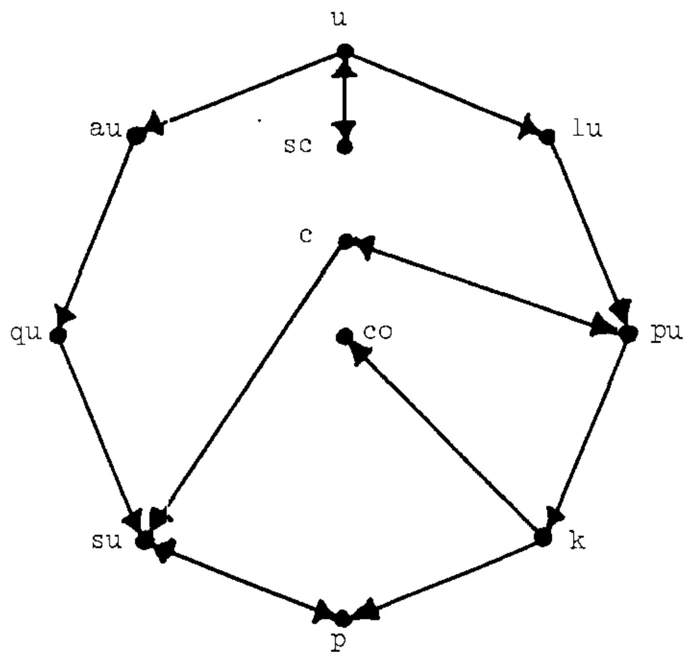

Figure 9. Y Compact, f_a Continuous

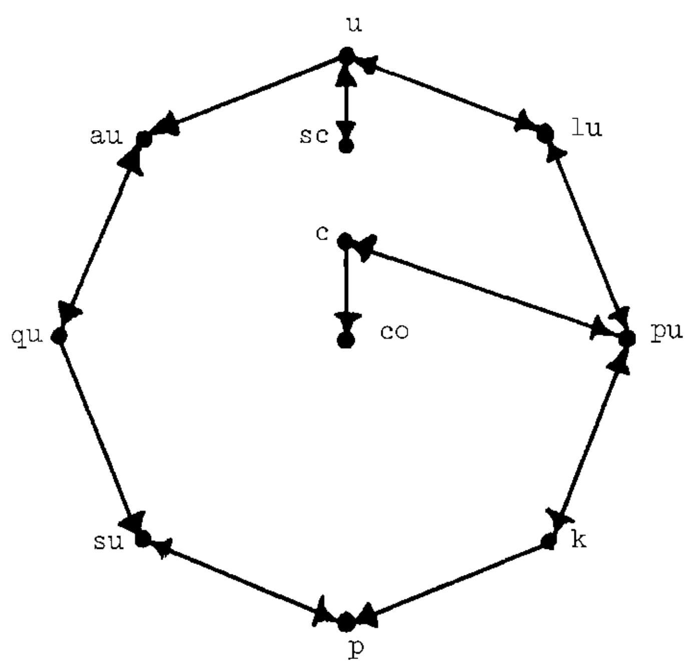

Figure 10. X and Y Compact, f_a Continuous

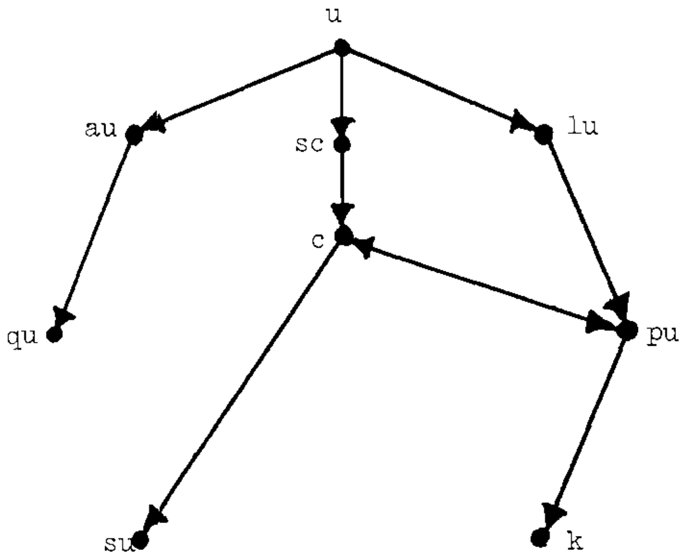

Figure 11. f Continuous

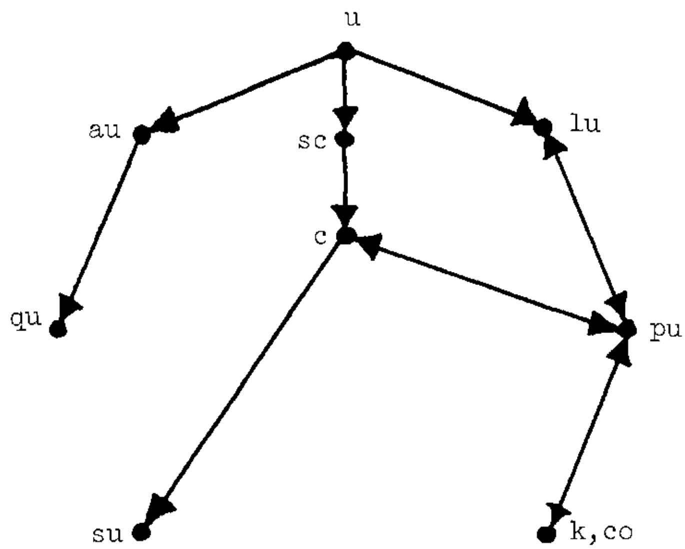

Figure 12. X Locally Compact, f Continuous

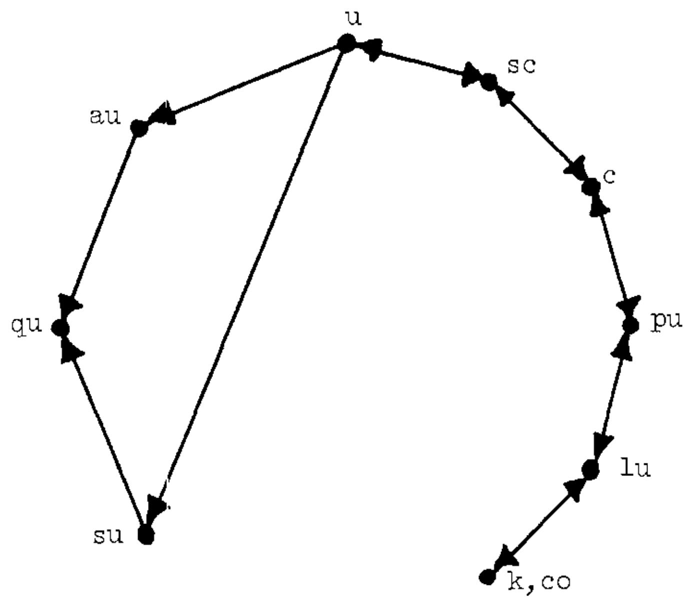

Figure 13. X Compact, f Continuous

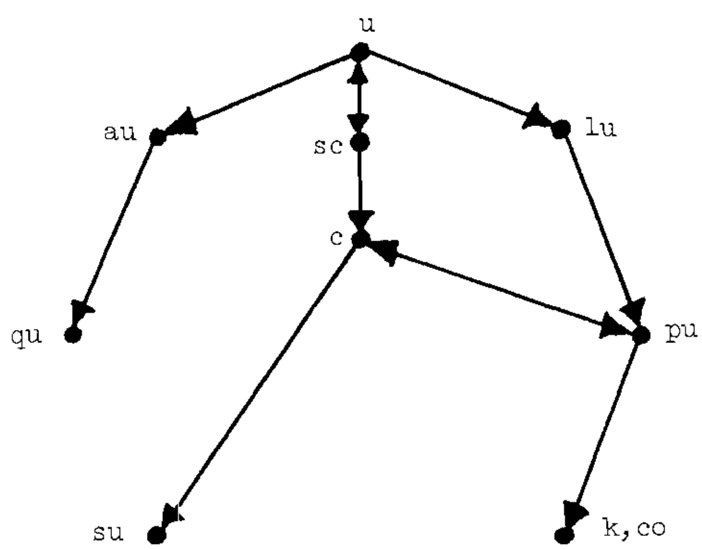

Figure 14. Y Compact, f Continuous

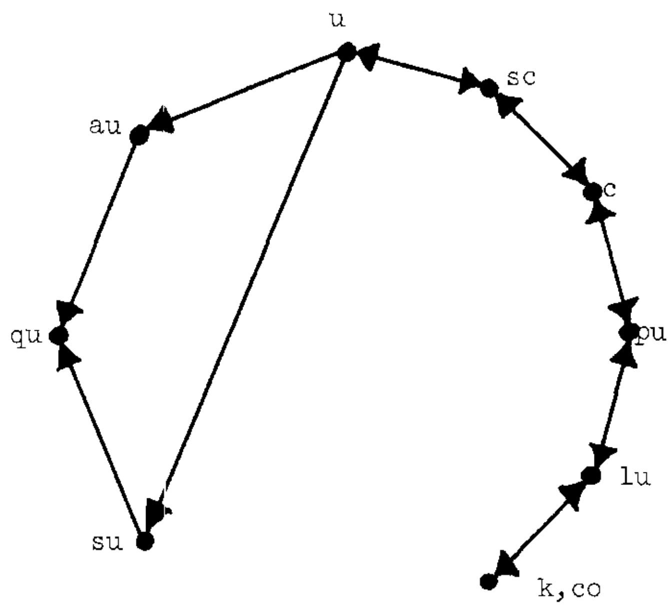

Figure 15. X and Y Compact, f Continuous

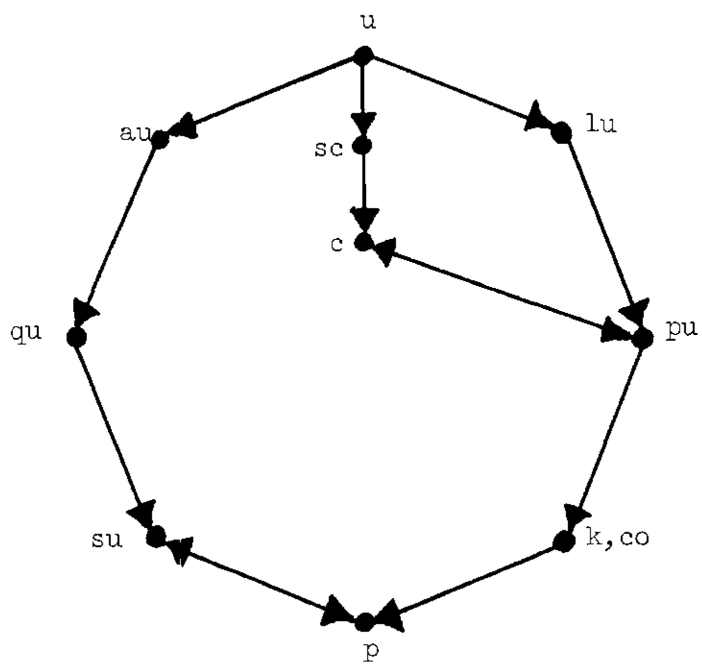

Figure 16. f_a and f Continuous

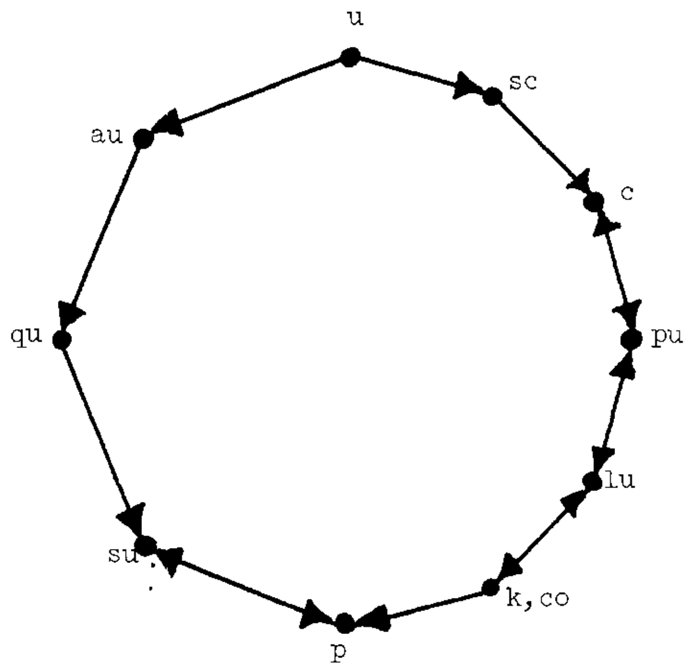

Figure 17. X Locally Compact, f_a and f Continuous

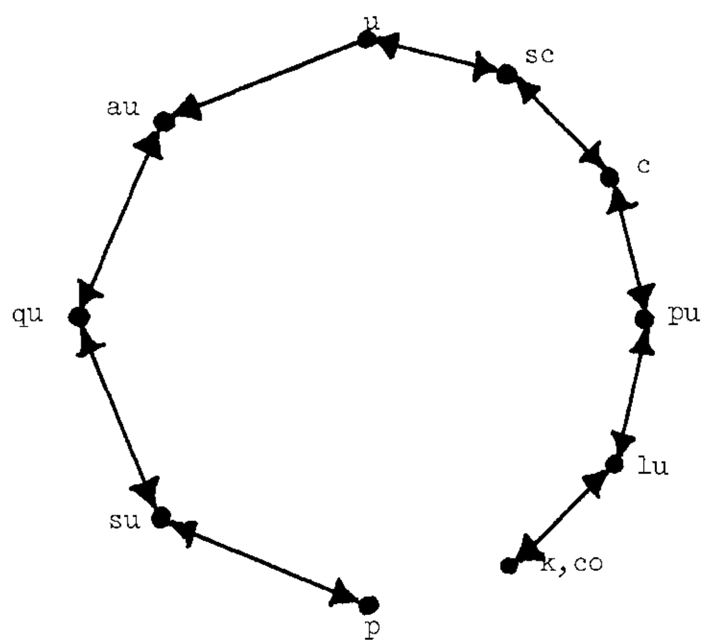

Figure 18. X Compact, f_a and f Continuous

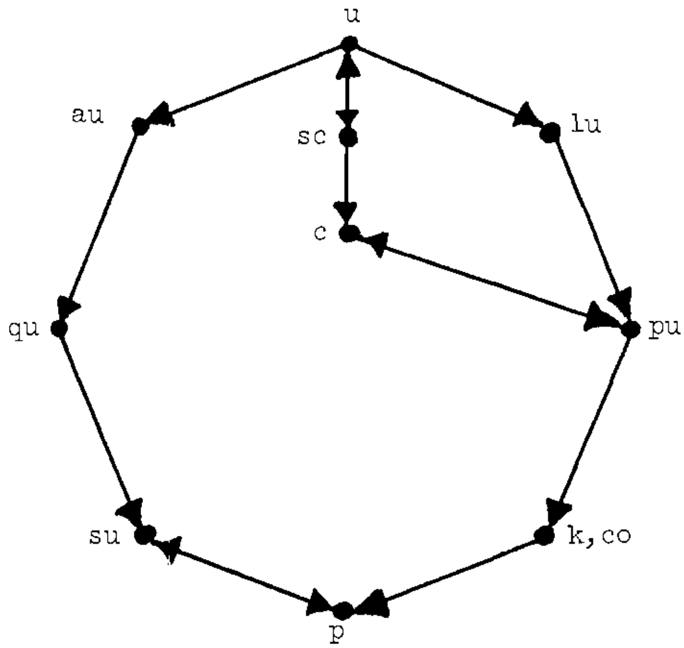

Figure 19. Y Compact, f_a and f Continuous

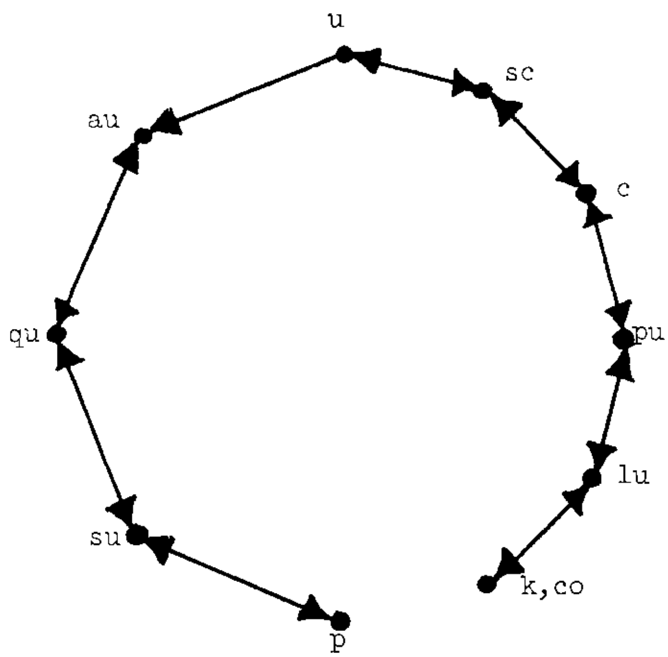

Figure 20. X and Y Compact, f_a and f Continuous

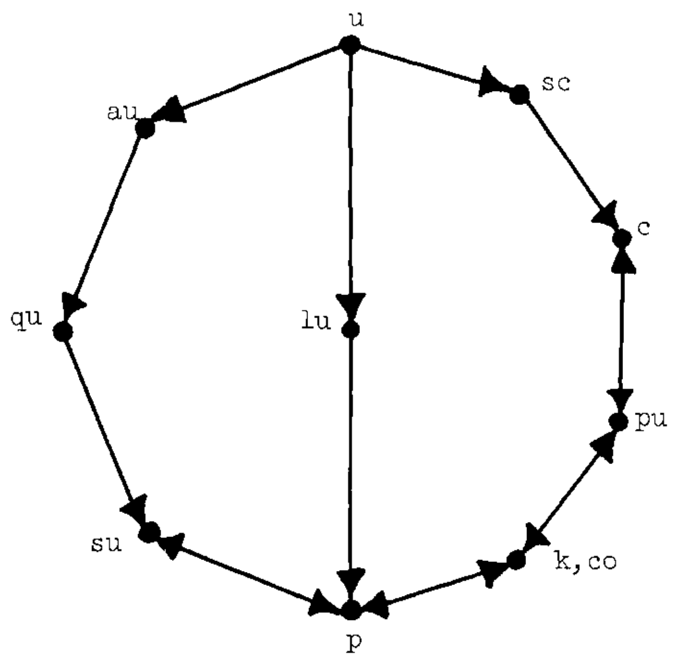

Figure 21. $\{f_a\}$ Evenly Continuous

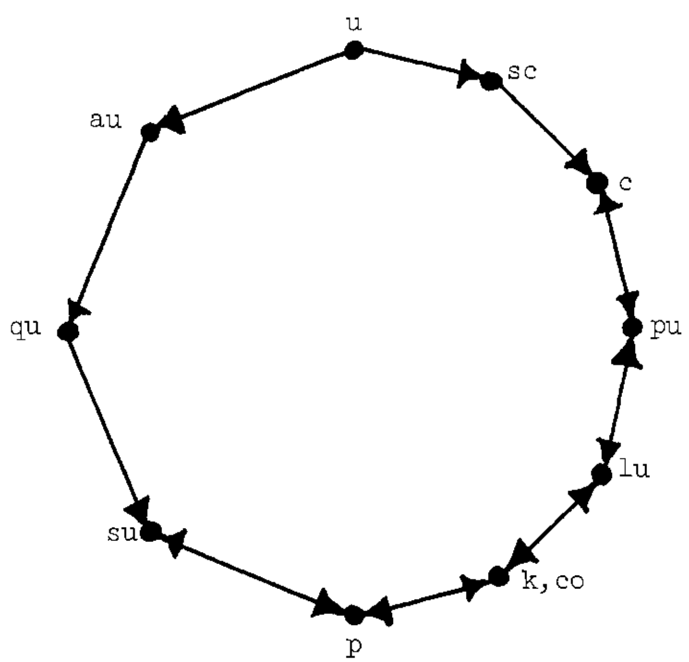

Figure 22. X Locally Compact, $\{f_a\}$ Evenly Continuous

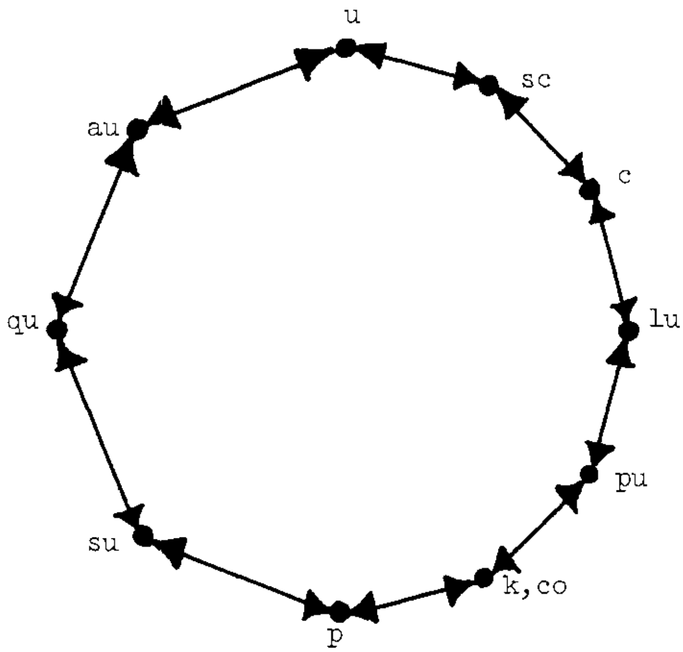

Figure 23. X Compact, $\{f_a\}$ Evenly Continuous

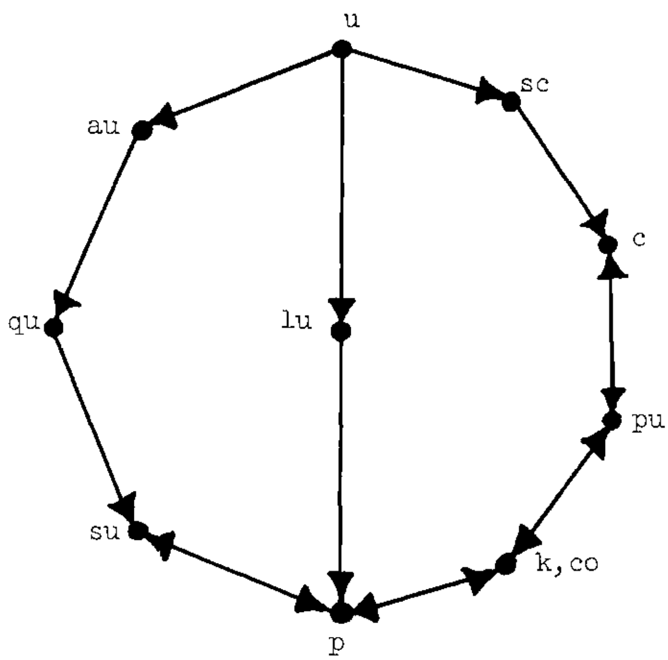

Figure 24. $\{f_a\}$ Uniformly Equicontinuous

4. Verifications of the Implications

For Figure 1.

X is assumed to be a topological space, Y a uniform topological space. Each f_a and f are assumed to be functions from X into Y.

(4.1) Lemma: If f_a converges (c) to f then f is continuous.

Proof: See Fuller [13, Theorem 7.6].

(4.2) Theorem: If f_a converges (u) to f then it converges (au) to f.

Proof: Let $\{f_{a_b}\}$ be a subnet of $\{f_a\}$. The net f_{a_b} converges (u) to f. There is an index N so that if $b \geq N$ then $(f_{a_b}(x), f(x))$ is in V for all x in X. Hence f_{a_b} converges (qu) to f from which it follows the original net converges (au) to f.

(4.3) Theorem: If f_a converges (au) to f then it converges (qu) to f.

Proof: By definition.

(4.4) Theorem: If f_a converges (pu) to f then it converges (k) to f.

Proof: See Fuller [13, Theorem 8.9].

(4.5) Theorem: If f_a converges (lu) to f then it converges (pu) to f.

Proof: Let x be in X. There exists a neighborhood U of x such that f_a converges (u) to f on U. This U works for all V in $\mathcal{V}$ in the definition of the convergence "pu".

(4.6) Theorem: If f_a converges (c) to f then it converges (pu) to f.

Proof: From Lemma (4.1) it follows that f is continuous. Let V be in $\mathcal{V}$, x in X, and W in $\mathcal{V}$ with $W \circ W \subset V$. There is a neighborhood U_1 of x such that $f(U_1) \subset W[f(x)]$. There is a neighborhood U_2 of x and an index n so that if $a \geq n$ and y is in U_2 then $(f(x), f_a(y))$ is in W^{-1}. Let $U = U_1 \cap U_2$. For y in U it follows that $(f(x), f(y))$ is in W. If $a \geq n$ then $(f_a(y), f(x))$ is in W so that $(f_a(y), f(y))$ is in $W \circ W \subset V$.

Remark: Poppe [18, Theorem 1.2] claims that the preceding theorem follows from an adaptation of a theorem in Hahn [14].

(4.7) Theorem: If f_a converges (u) to f then it converges (sc) to f.

Proof: Suppose that $f(x_b)$ converges to y. Let V be in $\mathcal{V}$. There is a W in $\mathcal{V}$, indices m and n such that $W \circ W \subset V$, and $a \geq m$ implies $(f_a(x), f(x))$ is in W for all x, and $b \geq n$ implies $(f(x_b), y)$ is in W. Hence if $a \geq m$ and $b \geq n$ then $(f_a(x_b), f(x_b))$ is in W and $(f(x_b), y)$ is in W. Therefore $(f_a(x_b), y)$ is in V.

(4.8) Theorem: If f_a converges (u) to f then it converges (lu) to f.

Proof: By definition.

(4.9) Theorem: If f_a converges (c) to f then it converges (su) to f.

Proof: Suppose f_a converges (c) to f. Let x be in X, V be in $\mathcal{V}$, and m be an index. There is a neighborhood U of x, and an index n, such that if $a \geq n$ and y is in U then $(f(x), f_a(y))$ is in V. If $a \geq m$ and $a \geq n$ then $(f_a(y), f(x))$ is in V whenever y is in U.

(4.10) Lemma: Let D be a compact subset of $(Y, \mathcal{V})$, and let U be an open set in Y containing D. There is a member T of $\mathcal{V}$ so that if y is in D and (y,y') is in T then y' is in U.

Proof: The proof is found in Kelley [15, p. 199.].

(4.11) Therem: If f is continuous then f_a converges (co) to f if and only if it converges (k) to f.

Proof: Suppose the convergence is "co". Let K be a compact subset of X. Let V be in $\mathcal{V}$. There is a W in $\mathcal{V}$ such that $W \circ W \circ W \circ W \subset V$. $\{W[f(x)]: x \text{ is in } K\}$ covers the compact set $f(K)$. There is a finite subcover $\{W[f(x_k)]: k = 1,2,\ldots,n\}$. Let $K_i = K \cap f^{-1}(\overline{W[f(x_i)]})$, for $i = 1,2,\ldots,n$. There is an index m so that if $a \geq m$ then $f_a(K_i) \subset W \circ W \circ [f(x_i)]$. It follows that if x is in K then there is an integer i so that x is in K_i and therefore $f_a(x)$ is in $W \circ W\, [f(x_i)]$. It follows that $(f_a(x), f(x_i))$ is in $W \circ W$, $f(x)$ is in $W \circ W\, [f(x_i)]$, and therefore $(f(x), f(x_i))$ is in $W \circ W$. Thus $(f_a(x), f(x))$ is in V.

Conversely suppose that K is a compact subset of X and U is an open subset of Y with $f(K) \subset U$. By Lemma (4.10) there is a T in $\mathcal{V}$ such that if y is in $f(K)$ and (y,y') is in T then y' is in U. There is an index m so that if $a \geq m$ then $(f(x), f_a(x))$ is in T for all x in K. Therefore $f_a(x)$ is in U for all x in K, i.e. $f_a(K) \subset U$ for all $a \geq m$.

(4.12) Theorem: If f_a converges (c) to f then it converges (co) to f.

Proof: The function f is continuous by Lemma (4.1). Theorem (4.11) together with Theorem (4.4) and Theorem (4.6) yield the result.

For Figure 2

The space X is assumed to be a locally compact topological space.

(4.13) Theorem: If f_a converges (k) to f then it converges (lu) to f.

Proof: By definition.

For Figure 3

The space X is assumed to be a compact topological space.

(4.14) Theorem: If f_a converges (k) to f then it converges (u) to f.

Proof: By definition.

For Figure 4

The space Y is assumed to be a compact topological uniform space.

(4.15) Theorem: If f_a converges (sc) to f then it converges (u) to f.

Proof: Suppose the net does not converge uniformly to f. There is a member V of $\mathcal{U}$ such that for each index n there is an index $a_n \geq n$ and a member x_{a_n} of X for which $(f_{a_n}(x_{a_n}), f(x_{a_n}))$ is not a member of V. Since Y is compact there is a subnet $\{f(x_{a_{n_j}})\}$ converging to some y in Y. Hence $f_a(x_{a_{n_j}})$ converges to y. Hence there is a symmetric W in $\mathcal{U}$ with $W \circ W \subset V$, indices a' and j' such that $(f_a(x_{a_{n_j}}), y)$ is in W when $a \geq a'$ and $j \geq j'$. There is a j'' such that if $j \geq j''$ then $(f(x_{a_{n_j}}), y)$ is in W. So, for $a \geq a'$, $j \geq j'$ and $j \geq j''$, it follows that $(f_a(x_{a_{n_j}}), f(x_{a_{n_j}}))$ is in V. But there is an index $j \geq j'$ and $j \geq j''$ such that $a_{n_j} \geq a'$ and thus $(f_{a_{n_j}}(x_{a_{n_j}}), f(x_{a_{n_j}}))$ is not in V, a contradiction.

For Figure 5

The spaces X and Y are assumed to be compact.

The arrows may be obtained by combining the results of Figure 3 and Figure 4.

For Figure 6

Each f_a is assumed to be continuous.

(4.16) Theorem: If f_a converges (qu) to f then f is a continuous function.

Proof: Let x be in X, V be in $\mathcal{U}$. There is a symmetric W in $\mathcal{U}$ such that $W \circ W \circ W \subset V$. There is an index m such that if $a \geq m$ then $(f_a(x), f(x))$ is in W. There are indices $a_1, \ldots, a_n$ each following m so that for each y in X there is at least one integer i so that $(f_{a_i}(y), f(y))$ is in W. Let U_i be the set $f_{a_i}^{-1}(W[f(x)])$. Let $U = \bigcap_{i=1}^{n} U_i$, and let y be in U. Then $(f_{a_i}(y), f(y))$ is in W for some i, $(f_{a_i}(y), f \quad (x))$ is in W so that $(f(x), f(y))$ is in $W \circ W \subset V$ and thus $f(U) \subset V [f(x)]$.

Remark: The preceding theorem is known as part of Arzela's theorem.

(4.17) Theorem: If f_a converges (p) to f then it converges (su) to f.

Proof: Let x be in X, V be in $\mathcal{U}$, and the index m be given. There is a symmetric W in $\mathcal{U}$ such that $W \subset V$. There is an index $a \geq m$ such that $f_a(x)$ is in $W[f(x)]$. Let $U = f_a^{-1}(W[f(x)])$. It follows that if y is in U then $f_a(y)$ is in $W[f(x)]$ and hence $(f_a(y), f(x))$ is in V.

(4.18) Theorem: If f_a converges (qu) to f then it converges (su) to f.

Proof: From Theorem (4.17) "su" convergence follows from "p" convergence.

(4.19) Lemma: If f_a converges (pu) to f then f is continuous.

Proof: Let x be in X, V be in $\mathcal{U}$, and let W be in $\mathcal{U}$ with $W \circ W \circ W \subset V$. There is an index m such that if $a \geq m$ then $(f_a(x), f(x))$ is in W. There is an index n and a neighborhood U' of x so that if $a \geq n$ and y is in U' then $(f_a(y), f(y))$ is in W. Let $a \geq m$ and $a \geq n$. There is a neighborhood U'' of x so that if y is in U'' then $(f_a(y), f_a(x))$ is in W. Let $U = U' \cap U''$. Now if y is in U it follows that $(f_a(y), f(y))$ is in W, $(f_a(y), f_a(x))$ is in W and $(f_a(x), f(x))$ is in W. Therefore $(f(y), f(x))$ is in $W \circ W \circ W \subset V$ and thus $f(U) \subset V[f(x)]$.

(4.20) Theorem: Let $\{f_a\}$ be a net of functions from X into Y with the f_a's not necessarily being continuous. If f_a converges (pu) to f and f is continuous then f_a converges (c) to f.

Proof: Let x be in X and V be in $\mathcal{U}$. There is a W in $\mathcal{U}$ with $W \circ W \subset V$. There is a neighborhood U' of x such that $f(U') \subset W[f(x)]$. There is a neighborhood U'' and an index m so that if $a \geq m$ and y is in U'' then $(f_a(y), f(y))$ is in W^{-1}. Let $U = U' \cap U''$. If $a \geq m$ and y is in U then $(f(y), f_a(y))$ is in W and $(f(x), f(y))$ is in W, whence $(f(x), f_a(y))$ is in V.

Remark: The preceding result is found in Fuller [13, Theorem 8.7] and Poppe [18, Theorem 1.2].

(4.21) Theorem: If f_a converges (k) to f then it converges (co) to f.

Proof: Let K be a compact subset of X and let U be an open set in Y for which $f(K) \subset U$. Now $f|_K$ is continuous and hence $f(K)$ is compact in the relative topology and thus compact in the space Y. By Lemma (4.10) there is a T in $\mathcal{V}$ so that if y is in $f(K)$ and (y,y') is in T then y' is in U. There is an index m such that if $a \geq m$ then $(f_a(x), f(x))$ is in T for each x in K. Hence $f_a(K) \subset U$ for all $a \geq m$.

For Figure 7

The space X is assumed to be locally compact and each f_a is assumed to be continuous. The implications may be obtained from a combination of Figure 2 and Figure 6.

For Figure 8

The space X is assumed to be compact and each f_a is assumed to be continuous. The implications may be obtained by combining the results of Figures 3 and 6 with Theorems (2.2) and (4.16).

For Figure 9

Y is assumed to be compact and each f_a is assumed to be continuous. The implications may be obtained by combining the results of Figure 4 and Figure 6.

For Figure 10

X and Y are assumed to be compact and each f_a is assumed to be continuous. The implications may be obtained by combining the results of Figure 5 and Figure 6.

For Figure 11

The function f is assumed to be continuous.

Remark: The fact that co = k follows from Theorem (4.11).

(4.22) Theorem: If f_a converges (pu) to f then it converges (c) to f.

Proof: See Theorem (4.20).

(4.23) Theorem: If f_a converges (sc) to f then it converges (c) to f.

Proof: Let x be in X and V be in $\mathcal{V}$. Direct the set $D = \{(y,U)$: U is a neighborhood of x and y is in $U\}$ by $(y,U) \geq (y',U')$ if and only if $U \subset U'$. For each (y,U) in D let $z_{(y,U)} = y$. The net $z_{(y,U)}$ converges to x and hence the net $f(z_{(y,U)})$ converges to $f(x)$. It follows that the net $f_a(z_{(y,U)})$ converges to $f(x)$. There is an index a' and a neighborhood U' of x so that $f_a(z_{(y,U)})$ is in $V[f(x)]$ whenever $a \geq a'$ and $U \subset U'$. Consequently $f_a(U') \subset V[f(x)]$.

Remark: Poppe [18, Theorem 1.2] has a version of the above for convergence spaces.

For Figure 12

The space X is assumed to be compact and f is assumed to be continuous. The implications may be derived by the combination of the results of Figure 2 and Figure 11.

For Figure 13

The space X is assumed to be compact and f is assumed to be continuous.

(4.24) Theorem: If f_a converges (su) to f then it converges (qu) to f.

Proof: Let V be in $\mathcal{U}$ and m be an index. There is a symmetric W in $\mathcal{U}$ for which $W \circ W \subset V$. Since the convergence is simply uniform there is for each x in X a neighborhood U'_x of x and an index $a_x \geq m$ with the property that if y is in U'_x then $(f_{a_x}(y), f(x))$ is in W. Since f is continuous there is a neighborhood U''_x of x such that $f(U''_x) \subset W[f(x)]$. Let $U_x = U'_x \cap U''_x$. There is subcover $\{U_{x_i} : i = 1,2,\ldots,n\}$ of X. If x is in X then x is in U_{x_i} for some integer i. For that i, $(f_{a_{x_i}}(x), f(x_i))$ is in W and $(f(x_i), f(x))$ is in W. Therefore $(f_{a_{x_i}}(x), f(x))$ is in V.

The remaining implications follow from the application of Figure 3 and Figure 11.

For Figure 14

It is assumed that the space Y is compact and that f is continuous. The implications may be derived from Figure 4 and Figure 11.

For Figure 15

Note that Figure 15 is the same as Figure 13.

For Figure 16

It is assumed that each f_a and f are continuous. The implications may be derived from Figure 6 and Figure 11.

For Figure 17

It is assumed that X is locally compact, each f_a is continuous and that f is continuous. The figure may be obtained by combining the results of Figure 2 and Figure 16.

For Figure 18

It is assumed that X is compact, each f_a is continuous and that f is continuous.

(4.25) Theorem: If f_a converges (p) to f then it converges (au) to f.

Proof: See Theorem (4.17) and Figure 13. The remaining implications follow from Figure 13.

For Figure 19

The space Y is assumed to be compact, f_a is assumed to be continuous and f is assumed to be continuous. The implications are a combination of those in Figure 16 and Figure 4.

For Figure 20

Both X and Y are assumed to be compact. Each f_a and f are assumed to be continuous. Note that Figure 20 is the same as Figure 18.

For Figure 21

The net $\{f_a\}$ is assumed to be evenly continuous.

(4.26) Lemma: If F is an evenly continuous family of functions from X into Y and $G \subset F$ then G is an evenly continuous family. Each f in F must be continuous.

Proof: The proof follows from definitions.

(4.27) Lemma: If the net $\{f_a\}$ is evenly continuous and if f_a converges (p) to f then f_a converges (c) to f, and f is continuous.

Proof: Let x be in X and V be a neighborhood of $f(x)$. There are neighborhoods W of $f(x)$ and U of x such that if a is any index for which $f_a(x)$ is in W then $f_a(U) \subset V$. But $f_a(x)$ converges (p) to $f(x)$. Therefore there is an index a' such that if $a \geq a'$ then $f_a(x)$ is in W. Hence for $a \geq a'$ it follows that $f_a(U) \subset W$. Continuity of f follows from Lemma (4.1).

The remainder of the implications, in view of Lemma (4.26), follow from Figure 16.

For Figure 22

It is assumed that X is locally compact and the net $\{f_a\}$ is evenly continuous. The implications may be found by combining Figure 2 and Figure 21.

For Figure 23

It is assumed that X is compact and the net $\{f_a\}$ is evenly continuous. The implications, in view of Lemma (4.26) and Lemma (4.27) are obtained by a combination of Figure 18 and Figure 21.

For Figure 24

Note that Figure 24 is the same as Figure 21.

5. Counter-Examples

Most of the examples concern functions from a subset of the real numbers with the usual topology to a subset of the real numbers with the usual topology. Unless otherwise specified a subset of the reals will have the usual topology. The following "roof top" function is useful in describing the counter-examples.

Definition: The function S is defined from R^3 into the real numbers by

$$S(a,b,x) = 1 - \left|\frac{2}{b-a}\left(x - \frac{a+b}{2}\right)\right|$$

whenever $a \leq x \leq b$. The function S is to be zero otherwise.

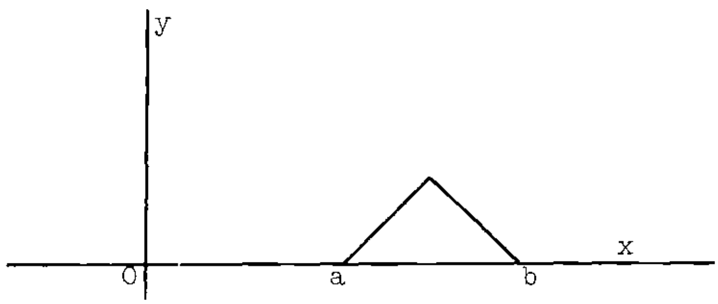

Figure 25. The Function S for Fixed a and b.

(5.1) Example: $X = Y = [0,2]$. Let $f_n(x) = f(x) = 1$ for x in $[0,1)$ and $f_n(x) = f(x) = 0$ on $[1,2]$. Both X and Y are compact. The convergence is co, sc, and u but not su.

(5.2) Example: $X = [0,1]$, $Y = (0,1]$. Let $f(x) = x$ for all $x \neq 0$ and $f(0) = 1$. Let $f_n(x) = x + S(-1/n, 1/n, x)$. The space X is compact while Y is locally compact. The convergence is sc but is neither k, qu nor co. The f_n's are continuous.

(5.3) Example: $X = Y = R$. Let $f(x) = x$ and $f_n(x) = (1 - 1/n)x$ for all real x. Both X and Y are locally compact. The convergence is lu, su, and sc but not qu. The sequence is uniformly equicontinuous and the function f is continuous.

(5.4) Example: $X = Y = R$. Let $f(x) = 0$ for all real x and $f_n(x) = x/n$ for all real x. Both X and Y are locally compact. Each f_a and f are continuous. The convergence is k, lu, and c but is neither qu nor sc.

(5.5) Example: $X = Y$ = set of all real null sequences. $\|x\| = \sup \{|x(n)| : n = 1,2,\ldots\}$. Let $f_n(x)(k) = 0$ for $k < n$ and $f_n(x)(k) = x(k)$ if $k \geq n$. Let $f(x)$ = the sequence of zeros. The sequence of continuous functions converges (c) but not (lu) to the continuous limit f.

Remark: The preceding example appears in Fuller [13, Example 8.8]. Example 4 appears in Poppe [18, Bemerkungen 1].

(5.6) Example: Define the function x_r from the unit interval into the unit interval by $x_r(t) = 1$ if $r = t$ and $x_r(t) = 0$ if $r \neq t$ for every r in $(0,1]$, and every t in $[0,1]$. Let X be the set of the functions x_r together with the zero function. Let Y be the closed unit interval. Let X have the topology of pointwise convergence. Note: Only finite sets are compact since each $\{x_r\}$ is an open set. Direct the set $D = \{r: r \text{ is in } (0,1]\}$ by $\leq$. For each r in D let $f_r(x) = x(r)$ for each x in X. The net f_r converges to the zero function, uniformly on compacta. Note that every neighborhood of the zero function contains an x_r frequently. Y is compact. The net of continuous functions converges (k) but not (pu) to the continuous limit f.

(5.7) Example: Let $X = Y = [0,1]$. Let $\{A_1, A_2, \ldots\}$ be a countable collection of pairwise disjoint dense subsets of $[0,1]$. Let $f(x) = 0$ for all x in X, $f_n(x) = 1$ for x in A_n and $f_n(x) = 0$ otherwise. Both X and Y are compact. The function f is continuous. The convergence is qu and au but is neither sc, k, nor su.

(5.8) Example: Let $X = Y = [0,1]$. Let $f_n(x) = S(-1/n, 1/n, x)$ for n even and $x > 0$, $f_n(0) = 0$ for n even, $f_n(x) = 0$ for n odd and $f(x) = 0$ for all x in X. Both X and Y are compact. The function f is continuous. The convergence is su and qu but not au.

(5.9) Example: Let $X = (0,1]$, $Y = [0,1]$, $f_n(x) = S(-1/n, 1/n, x)$ and $f(x) = 0$ for all x in X. The space X is locally compact and Y is compact. The evenly continuous sequence converges (lu) but not (qu) to the continuous limit f.

(5.10) Example: Let x denote a real valued sequence whose kth term is $x(k)$ and let $\|x\| = \sup\{x(k): k = 1,2,\ldots\}$. Let $(X, \|\ \|)$ be the normed linear space of all bounded real valued sequences. Define the double sequence $\{r_{ni}\}$ in X by

$$r_{ni}(k) = 1/n \text{ if } k = 1$$

$$r_{ni}(k) = 0 \text{ if } k > 1 \text{ and } k \neq i$$

$$r_{ni}(k) = 1/(3n^2) \text{ if } k > 1 \text{ and } k = i.$$

For $n, i = 1,2,\ldots$ let

$$m_{ni}(x) = \max\{1/(9n^2) - \|x - r_{ni}\|, 0\}.$$

For $i = 1,2,\ldots$ define the function f_i from X into X by: For $k = 1,2,\ldots$ and for each x in X let

$$f_i(x)(k) = x(k) + \sum_{n=1}^{\infty} m_{ni}(x).$$

We will show that for each x in X at most finitely many of the $m_{ni}(x)$'s are non zero. Define the function f from X into X by $f(x) = x$ for each x in X. The sequence $\{f_i\}$ converges strongly continuously and almost uniformly to f but does not converge locally uniformly to f. The family $\{f_i: i = 1,2,\ldots\}$ is uniformly equicontinuous.

Due to the complexity of the example a verification of the assertions is in order.

(i) We show first that for each x in X there is at most one of the $m_{ni}(x)$'s which is non zero. To that end we show that if $(n,i) \neq (k,j)$ then $\| r_{ni} - r_{kj} \| \geq \max\{1/(3k^2), 1/(3n^2)\}$. If $n \neq k$ clearly $\| r_{ni} - r_{kj} \| = |1/k - 1/n|$. One can show that $|1/k - 1/n| \geq \max\{1/(3k^2), 1/(3n^2)\}$ when k and n are different integers. If $k = n$ and $i \neq j$ then $\| r_{ni} - r_{kj} \| = 1/(3n^2) = 1/(3k^2) = \max\{1/(3k^2), 1/(3n^2)\}$. Finally suppose that for some x we have both $m_{ni}(x) \neq 0$ and $m_{kj}(x) \neq 0$ with $(n,i) \neq (k,j)$. It follows that both $\| x - r_{kj} \| \leq 1/(9k^2)$ and $\| x - r_{ni} \| \leq 1/(9n^2)$. Hence $\| r_{ni} - x + x - r_{kj} \| \leq 1/(9n^2) + 1/(9k^2)$ and consequently $\| r_{ni} - r_{kj} \| \leq (2/3) \max\{1/(3n^2), 1/(3k^2)\}$ which contradicts the fact that $\| r_{ni} - r_{kj} \| \geq \max\{1/(3n^2), 1/(3k^2)\}$.

(ii) Next we will show that f_i converges strongly continuously to f. Let $\{x_b\}$ be a net in X for which $f(x_b)$ converges to y in X. Since $f(x) = x$ we have x_b converges to y. We must show that the net

$S(i,b) = f_i(x_b)$ converges to y.

First suppose $y \neq \overline{0}$, where $\overline{0}$ denotes the null sequence. Let $\epsilon > 0$ be given. We will show there is a neighborhood N of y such that $N \cap \{t: \|t - r_{ni}\| < 1/(9n^2)\} \neq \emptyset$ for at most a finite number of r_{ni}'s. Since $y \neq \overline{0}$ it follows that $y(k) \neq 0$ for some integer k. Choose a positive integer n such that $4/(3n) < |y(k)/2|$. Let $m > n$ and let x be in the set $\{t: \|t - r_{mi}\| < 1/(9m^2)\}$. We have the following inequalities: $\|x - r_{mi}\| < 1/(9m^2) < 1/(3m) < 1/(3n)$ and $\|r_{mi}\| = 1/m < 1/n$. Also $\|x\| = \|x - r_{mi} + r_{mi}\| < 1/(3n) + 1/n = 4/(3n) < |y(k)/2|$. We assert that $\|x - y\| \geq |y(k)/2|$, for suppose that $|x(k) - y(k)| < |y(k)/2|$. We would then have $|y(k)| = |y(k) - x(k) + x(k)| \leq |y(k) - x(k)| + |x(k)|$ and hence $|y(k)| < |y(k)/2| + \|x\| < |y(k)/2| + |y(k)/2| = |y(k)|$, a contradiction. Hence $|y(k)/2| \leq |x(k) - y(k)| \leq \|x - y\|$, when x is in $\{t: \|t - r_{mi}\| < 1/(9m^2)\}$. Let $N = \{x: \|x - y\| < \min\{|y(k)/2|, 1/(18n^2)\}$. The preceding inequalities establish that $N \cap \{t: \|t - r_{mi}\| < 1/(9m^2)\} = \emptyset$ for all $m \geq n$. Hence the preceding intersection can be non-empty only for $m < n$. We now show that for each $m = 1,2,\ldots,n - 1$ there is at most one i for which the preceding intersection is non empty. Suppose for some $m < n$ and integer i that x is in $N \cap \{t: \|t - r_{mi}\| < 1/(9m^2)\}$. Recall that $r_{mi}(i) = 1/(3m^2)$. We have that $|x(i) - 1/(3m^2)| = |x(i) - r_{mi}(i)| \leq \|x - r_{mi}\| < 1/(9m^2)$ and $|x(i) - y(i)| \leq \|x - y\| < 1/(18n^2) < 1/(18m^2)$. Hence $|1/3m^2) - y(i)| = |1/(3m^2) - x(i) + x(i) - y(i)| < 1/(9m^2) + 1/(18m^2)$. Thus $|1/(3m^2) - y(i)| < (3/(18m^2)$. It now follows that $N \cap \{t: \|t - r_{mj}\| < 1/(9m^2)\}$ must be $\emptyset$ when $j \neq i$. If not, there is a z in the preceding intersection for which

$|z(i) - y(i)| \leq \| z - y\| < 1/(18n^2) \leq 1/(18m^2)$. Also $|z(i) - r_{mj}(i)| =$ $|z(i)| \leq 1/(9m^2)$ since $r_{mj}(i) = 0$.

We would now have

$$1/(3m^2) = |1/(3m^2) - y(i) + y(i) - z(i) + z(i)|$$

and hence

$$1/(3m^2) \leq |1/(3m^2) - y(i)| + |y(i) - z(i)| + |z(i)|.$$

Therefore

$$1/(3m^2) < 3/(18m^2) + 1/(18m^2) + 1/(9m^2) = 1/(3m^2)$$

which is a contradiction. Hence for $j \neq i$ we have $N \cap \{t : \| t - r_{mj}\| < 1/(9m^2)\} = \emptyset$. Thus the intersection is non empty for at most one j for each $m < n$, and we have established that the intersection is non empty for at most a finite number of r_{ni}'s where $n, i = 1,2,\ldots$

The preceding result guarantees a positive integer i' such that if $i > i'$ then $N \cap \{t : \| t - r_{ni}\| < 1/(9n^2)\} = \emptyset$ for each $n = 1,2,\ldots$ So for $i > i'$ we have $f_i(x) = x$ for each x in N. There is an index b' such that if $b \geq b'$ then $x_b \epsilon N$. Since x_b converges to y there is an index b'' such that if $b \geq b''$ then $\| x_b - y\| < \epsilon$. Let b''' be an index for which $b''' \geq b''$ and $b''' \geq b'$. It follows that if $(i,b) \geq (i',b''')$ then $\| f_i(x_b) - y\| = \| x_b - y\| < \epsilon$. We have thus established that the net $f_i(x_b)$ converges to y when $y \neq \overline{0}$ and consequently f_i converges strongly continuously to $y \neq \overline{0}$.

Suppose now that $y = \overline{0}$ and that $f(x_b)$ converges to $\overline{0}$. It follows that x_b converges to $\overline{0}$. Let $\epsilon > 0$. There is an index b' such that if $b \geq b'$ then $\| x_b\| < \epsilon/2$. Let i and b with $b \geq b'$ be fixed. We have shown there is at most one n for which $\| x_b - r_{ni}\| < 1/(9n^2)$ since

$m_{ni}(x) \neq 0$ for at most one m_{ni}. For such an m_{ni} we have $\| x_b - r_{ni}\| < 1/(9n^2) < 1/(3n)$. It follows that $2/(3n) \leq \| x_b\|$. If not then $\| x_b\| < 2/(3n)$ and $\| r_{ni}\| = \| r_{ni} - x_b + x_b\| \leq \| r_{ni} - x_b\| + \| x_b\| < 1/(3n) + 2/(3n) = 1/n$ contradicting $\| r_{ni}\| = 1/n$. We now have for each $k = 1,2,\ldots$ $|f_i(x_b)(k)| = |x_b(k) + \max \{1/(9n^2) - \| x_b - r_{ni}\|, 0\}|$. Therefore $|f_i(x_b)(k) \leq |x_b(k)| + 1/(9n^2) \leq \| x_b\| + 1/(9n^2) < 2/(3n) + 1/(3n) = 1/n$. But $2/(3n) \leq \| x_b\| < \epsilon/2$ and therefore $1/n < (3/4)\,\epsilon < \epsilon$. Hence for $(i,b) \geq (1,b')$ we have shown that $|f_i(x_b)(k)| < \epsilon$ for each $k = 1,2,\ldots$ and therefore $\| f_i(x_b)\| < \epsilon$. Consequently the net $f_i(x_b)$ converges to $\overline{0}$ thereby completing the proof that f_i converges strongly continuously to f.

(iii) We will show here that f_i converges almost uniformly to f. Let the integer m and the positive number ϵ be given. Let $a_1 = m$ and $a_2 = m + 1$. Given an x in X either $f_{a_1}(x) = x$ or $f_{a_2}(x) = x$ since at most one of the $m_{ni}(x)$'s is non zero. Hence we have shown there are a_1, $a_2 \geq m$ such that for each x in X there is at least one a_i in $\{a_1, a_2\}$ such that $|f_{a_i}(x) - f(x)| = 0 < \epsilon$, i.e. f_i converges quasi-uniformly to f. The same argument can be applied to the subnets of $\{f_i\}$ and therefore f_i converges almost uniformly to f.

(iv) We will show next that f_i doesn't converge locally uniformly to f. Recall that $\| r_{ni}\| = 1/n$. Suppose there is a neighborhood U of $\overline{0}$ on which f_i converges to f uniformly. One can choose an integer n so that for each $i = 1,2,\ldots$ we have r_{ni} in U. If the convergence is uniform on U then there is an $N > 0$ so that if $i \geq N$ and x is in U then $\| f_i(x) - f(x)\| < 1/(9n^2)$. But r_{ni} is in U for each $i = 1,2,\ldots$

and for $k = 1,2,\ldots$ we have

$$f_i(r_{ni})(k) = r_{ni}(k) + 1/(9n^2) = f(r_{ni})(k) + 1/(9n^2).$$

Therefore

$$|f_i(r_{ni})(k) - f(r_{ni})(k)| = 1/(9n^2) \leq \| f_i(r_{ni}) - f(r_{ni})\|$$

which is a contradiction.

(v) We show finally that $\| f_i(x) - f_i(y)\| \leq 3\| x - y\|$ for each x, y in X and each positive integer i. To that end we first show that $|m_{ni}(x) - m_{ni}(y)| \leq \| x - y\|$, for each x, y in X and $n, i = 1,2,\ldots$ The proof is broken down into four cases.

Case 1. Suppose $m_{ni}(x) = m_{ni}(y) = 0$. The assertion follows immediately.

Case 2. Suppose $m_{ni}(x) \neq 0$ and $m_{ni}(y) = 0$. Here we have $\| y - r_{ni}\| \geq 1/(9n^2)$ and $\| x - r_{ni}\| < 1/(9n^2)$. Therefore $|m_{ni}(x) - m_{ni}(y)| = 1/(9n^2) - \| x - r_{ni}\|$. We must show that $1/(9n^2) - \| x - r_{ni}\| \leq \| x - y\|$. Suppose not. Then

$\| y - x + x - r_{ni}\| \leq \| y - x\| + \| x - r_{ni}\| < 1/(9n^2) - \| x - r_{ni}\| + \| x - r_{ni}\|$. Hence $\| y - r_{ni}\| < 1/(9n^2)$ which is a contradiction.

Case 3. Suppose $m_{ni}(x) = 0$ and $m_{ni}(y) \neq 0$. The result follows by interchanging the roles of x and y in case 2.

Case 4. Suppose $m_{ni}(x) \neq 0$ and $m_{ni}(y) \neq 0$. Here $|m_{ni}(x) - m_{ni}(y)| = |\| x - r_{ni}\| - \| y - r_{ni}\|| \leq \| x - y\|$.

The assertion in (v) now follows from four additional cases.

Case 1. Suppose $m_{ni}(x) = m_{ni}(y) = 0$ for $n = 1,2,\ldots$ Let $k = 1,2,\ldots$ For each k we have $|f_i(x)(k) - f_i(y)(k)| = |x(k) - y(k)| \leq \| x - y\|$ and hence $\| f_i(x) - f_i(y)\| \leq \| x - y\| \leq 3\| x - y\|$.

Case 2. Suppose $m_{pi}(x) \neq 0$ for some p and $m_{ni}(y) = 0$ for all $n = 1,2,\ldots$ Recall $m_{ni}(x) = 0$ for all $n \neq p$. We have $|f_i(x)(k) - f_i(y)(k)| = |x(k) - y(k) + m_{pi}(x) - m_{pi}(y)|$. Therefore $|f_i(x)(k) - f_i(y)(k)| \leq |x(k) - y(k)| + |m_{pi}(x) - m_{pi}(y)|$ and $|f_i(x)(k) - f_i(y)(k)| \leq \| x - y\| + \| x - y\| \leq 3\| x - y\|$. Thus $\| f_i(x) - f_i(y)\| \leq 3\| x - y\|$.

Case 3. Suppose $m_{pi}(x) \neq 0$ and $m_{qi}(y) \neq 0$ with $p \neq q$. Note that $m_{pi}(y) = m_{qi}(x) = 0$. Hence for $k = 1,2,\ldots$ we have

$|f_i(x)(k) - f_i(y)(k)| = |x(k) - y(k) + m_{pi}(x) - m_{pi}(y) + m_{qi}(x) - m_{qi}(y)|$

and consequently

$|f_i(x)(k) - f_i(y)(k)| \leq |x(k) - y(k)| + |m_{pi}(x) - m_{pi}(y)| + |m_{qi}(x) - m_{qi}(y)|$.

Therefore

$|f_i(x)(k) - f_i(y)(k)| \leq \|x - y\| + \| x - y\| + \| x - y\| = 3\| x - y\|$ and hence $\| f_i(x) - f_i(y)\| \leq 3\| x - y\|$.

Case 4. Suppose $m_{pi}(x) \neq 0$ and $m_{pi}(y) \neq 0$. Consequently $m_{ni}(x) = m_{ni}(y) = 0$ for all $n \neq p$. For $k = 1, 2,\ldots$ we have $|f_i(x)(k) - f_i(y)(k)| = |x(k) - y(k) + m_{pi}(x) - m_{pi}(y)|$ and $|f_i(x)(k) - f_i(y)(k)| \leq \| x - y\| + \| x - y\| \leq 3\| x - y\|$. Therefore $\| f_i(x) - f_i(y)\| \leq 3\| x - y\|$.

In all cases we have established the inequality

$$\| f_i(x) - f_i(y)\| \leq 3\| x - y\| .$$

It follows from the inequality that the family $\{f_i: \ i = 1,2,\ldots\}$ is uniformly equicontinuous (and therefore evenly continuous).

(5.11) Example: Let $X = Y = [0,1]$. Let $f_n(x) = S(1/(n + 1),\ 1/n,\ x)$ and $f(x) = 0$ for all x in X. Both X and Y are compact. The sequence is evenly continuous and f is continuous. The convergence is au and su but is neither sc, k, nor co.

(5.12) Example: This example is the same as Example 11 except that $X = (0,1]$. The space S is locally compact and Y is compact. The sequence is evenly continuous and f is continuous. The convergence is au and lu but not sc.

(5.13) Example: Let $X = Y = [0,1]$. Let $f(x) = 1$ for $x \neq 0$, $f(0) = 0$, $f_n(x) = 0$ for $0 \leq x \leq 1/n$, and $f_n(x) = 1$ for $1/n < x \leq 1$. Both X and Y are compact. The convergence is co but is neither k nor sc.

(5.14) Example: Let $X = Y = [0,1]$. For even integers n let $f_n(x) = S(-1/n,\ 1/n,\ x)$. For odd integers n let $f_n(x) = f(x) = 0$ for $x > 0$ and $f_n(x) = f(x) = 1$ for $x = 0$. Both X and Y are compact. The convergence is qu but not au.

(5.15) Example: Let $X = Y = [0,1]$. Let $f(x) = 0$ for $x > 0$, $f(0) = 1$, and $f_n(x) = S(-1/n,\ 1/n,\ x)$ for each x in X. Both X and Y are compact. The functions in the sequence are continuous. The convergence is su but is neither k, qu, nor sc.

(5.16) Example: Let $Y = [0,1]$. Let X be the set $[0,1] - \{1,\ 1/2,\ 1/3,\ldots\}$. Let $\{r_{nk}\colon\ k = 1,2,\ldots\}$ be an increasing sequence in $(1/(n + 1),\ 1/n)$ with $\lim_{k\to\infty} r_{nk} = 1/n$, for each $n = 1,2,\ldots$ Let $L(A,B,x)$

denote the y-coordinate of the point on the line segment connecting the points A and B in the plane at the value x on the x-axis. Define the function f_m for $m = 1,2,\ldots$ by

$$f_m(x) = x \text{ if } 1/(n+1) < x < r_{nm} \text{ and}$$

$$f_m(x) = L((r_{nm}, r_{nm}), (1/n, 0), x) \text{ if } r_{nm} \leq x < 1/n.$$

Let $f(x) = x$ for all x in X. The space Y is compact. The functions in the sequence and f are continuous. The convergence is su and c but is neither lu nor qu.

(5.17) Example: This example is same as Example 16 except that $Y = [0,1] - \{1, 1/2, 1/3, \ldots\}$. The net of continuous functions converges (sc) but not (lu) to the continuous limit f.

(5.18) Example: Let $X = (0,1)$ and $Y = (0,\infty)$. Let $f_n(x) = 1/(nx)$, and $f(x) = 0$ for all x in X. Both X and Y are locally compact. The sequence of continuous functions converges (su) but not (qu) to the continuous limit f.

(5.19) Example: Let X be the real numbers and let the open sets be the complements of countable subsets (hence only the finite sets are compact). Let Y be the closed unit interval with the usual topology. Define the net $\{f_r\colon\ r > 0\}$ by $f_r(x) = 1$ if $x = r$ and $f_r(x) = 0$ if $x \neq r$ for all x in X. Direct the index set by $\leq$. Let $f(x) = 0$ for all x in X. Note that every neighborhood of 0 contains uncountably many r's and hence $f_r(x) = 1$ frequently in any neighborhood of 0. The space Y is compact. The convergence is k but is neither su nor lu. The function

f is continuous.

(5.20) Example: Let $X = [0,1]$, $Y = [0,2]$. Let $f(x) = x$ for $0 \leq x < 1$ and $f(1) = 0$. Let $f_n(x) = x + 1/n$ for $0 \leq x < 1$ and $f_n(1) = 0$. Both X and Y are compact. The convergence is u and sc but not co.

(5.21) Example: Let $X = [0, \infty)$ and $Y = [0,1]$. Let $f_n(x) = S(n, n + 1, x)$ for all x in X. Let $f(x) = 0$ for all x in X. The space X is locally compact and Y is compact. The sequence is uniformly equicontinuous and the function f is continuous. The convergence is lu and au but not sc.

(5.22) Example: Let $X = [0,\infty)$ and $Y = [0,1]$. Let $f_n(x) = 0$ for n even and $0 \leq x < n$, let $f_n(x) = x - n$ for n even and $n \leq x \leq n + 1$, let $f_n(x) = 1$ for n even and $x > n + 1$. Let $f_n(x) = 0$ for n odd and let $f(x) = 0$ for all x in X. The space X is locally compact and Y is compact. The sequence is uniformly equicontinuous and f is continuous. The convergence is qu but not au.

(5.23) Example: Let X consist of the reciprocals of the positive integers together with the number zero. Let $f(x)$ be 1 when $x \neq 0$ and x is in X. Let $f(0) = 0$. Let $f_n(x) = 1$ when x is in X and $1/n \leq x \leq 1$ and let $f_n(x) = 0$ when x is in X and $0 \leq x < 1/n$. Both X and Y are compact. The functions in the sequence are continuous. The convergence is co but is neither qu, k, nor sc.

Table 1. Counter-Examples for Figure 1.

Item	Purpose	Example Number
1	u $\not\to$ su	1
2	sc $\not\to$ k	2
3	sc $\not\to$ qu	3
4	c $\not\to$ qu	4
5	c $\not\to$ sc	4
6	c $\not\to$ lu	5
7	k $\not\to$ pu	6
8	au $\not\to$ sc	7
9	au $\not\to$ k	7
10	qu $\not\to$ au	22
11	su $\not\to$ sc	11
12	su $\not\to$ k	11
13	lu $\not\to$ sc	12
14	lu $\not\to$ qu	4
15	u $\not\to$ co	20
16	su $\not\to$ co	11
17	co $\not\to$ k	13
18	co $\not\to$ su	1

Table 2. Counter-Examples for Figure 2.
X Locally Compact

Item	Purpose	Example Number
1	u $\not\to$ su	1
2	au $\not\to$ k	7
3	au $\not\to$ sc	7
4	au $\not\to$ su	7
5	qu $\not\to$ au	8
6	sc $\not\to$ k	2
7	sc $\not\to$ qu	2
8	sc $\not\to$ su	1
9	c $\not\to$ qu	4

Table 2. Continued

Item	Purpose	Example Number
10	$c \not\to sc$	4
11	$su \not\to k$	15
12	$u \not\to co$	20
13	$su \not\to co$	11
14	$co \not\to k$	13
15	$co \not\to su$	1
16	$co \not\to sc$	13

Table 3. Counter-Examples for Figure 3.
X Compact

Item	Purpose	Example Number
1	$u \not\to su$	1
2	$sc \not\to qu$	2
3	$sc \not\to su$	1
4	$au \not\to sc$	7
5	$qu \not\to au$	8
6	$su \not\to qu$	15
7	$su \not\to sc$	15
8	$u \not\to co$	20
9	$su \not\to co$	11
10	$co \not\to k$	13
11	$co \not\to su$	1
12	$co \not\to sc$	13
13	$co \not\to qu$	23

Table 4. Counter-Examples for Figure 4.
Y Compact

Item	Purpose	Example Number
1	$u \not\to su$	1

Table 4. Continued

Item	Purpose	Example Number
2	au $\not\to$ k	11
3	qu $\not\to$ au	14
4	c $\not\to$ lu	16
5	c $\not\to$ qu	16
6	k $\not\to$ pu	6
7	su $\not\to$ k	11
8	u $\not\to$ co	20
9	su $\not\to$ co	11
10	co $\not\to$ k	13
11	co $\not\to$ su	1

Table 5. Counter-Examples for Figure 5.
X and Y Compact

Item	Purpose	Example Number
1	qu $\not\to$ au	14
2	su $\not\to$ qu	15
3	au $\not\to$ co	11
4	sc $\not\to$ co	20
5	co $\not\to$ su	1
6	co $\not\to$ k	23
7	co $\not\to$ sc	23
8	co $\not\to$ qu	23

Table 6. Counter-Examples for Figure 6.
f_a Continuous

Item	Purpose	Example Number
1	au $\not\to$ co	11
2	au $\not\to$ sc	11
3	qu $\not\to$ au	22

Table 6. Continued

Item	Purpose	Example Number
4	su $\not\rightarrow$ qu	18
5	sc $\not\rightarrow$ k	2
6	c $\not\rightarrow$ sc	4
7	c $\not\rightarrow$ lu	5
8	k $\not\rightarrow$ pu	6
9	c $\not\rightarrow$ qu	4
10	k $\not\rightarrow$ sc	4
11	lu $\not\rightarrow$ sc	12
12	lu $\not\rightarrow$ qu	4
13	sc $\not\rightarrow$ co	2
14	co $\not\rightarrow$ k	23
15	sc $\not\rightarrow$ qu	3

Table 7. Counter-Examples for Figure 7.
X Locally Compact and f_a Continuous

Item	Purpose	Example Number
1	au $\not\rightarrow$ co	11
2	au $\not\rightarrow$ sc	11
3	sc $\not\rightarrow$ co	2
4	c $\not\rightarrow$ sc	4
5	c $\not\rightarrow$ qu	4
6	qu $\not\rightarrow$ au	22
7	co $\not\rightarrow$ k	23
8	sc $\not\rightarrow$ qu	3

Table 8. Counter-Examples for Figure 8.
X Compact and f_a Continuous

Item	Purpose	Example Number
1	sc $\not\to$ au	2
2	au $\not\to$ co	11
3	sc $\not\to$ co	2
4	co $\not\to$ sc	23
5	au $\not\to$ sc	11
6	co $\not\to$ qu	23

Table 9. Counter-Examples for Figure 9.
Y Compact and f_a Continuous

Item	Purpose	Example Number
1	qu $\not\to$ au	22
2	c $\not\to$ lu	16
3	lu $\not\to$ qu	9
4	au $\not\to$ co	11
5	co $\not\to$ k	23
6	k $\not\to$ pu	6

Table 10. Counter-Examples for Figure 10.
X and Y Compact, f_a Continuous

Item	Purpose	Example Number
1	co $\not\to$ qu	23
2	au $\not\to$ co	11

Table 11. Counter-Examples for Figure 11.
f Continuous

Item	Purpose	Example Number
1	au $\not\to$ su	7
2	au $\not\to$ k	7
3	qu $\not\to$ au	22
4	qu $\not\to$ k	7
5	qu $\not\to$ su	7
6	su $\not\to$ qu	3
7	su $\not\to$ k	11
8	sc $\not\to$ qu	3
9	sc $\not\to$ lu	17
10	c $\not\to$ sc	4
11	k $\not\to$ su	19
12	lu $\not\to$ sc	12
13	lu $\not\to$ qu	4
14	k $\not\to$ lu	19

Table 12. Counter-Examples for Figure 12.
X Locally Compact and f Continuous

Item	Purpose	Example Number
1	au $\not\to$ su	7
2	qu $\not\to$ au	22
3	sc $\not\to$ qu	3
4	c $\not\to$ sc	4
5	su $\not\to$ k	11

Table 13. Counter-Examples for Figure 13.
X Compact and f Continuous

Item	Purpose	Example Number
1	au $\not\to$ su	7
2	qu $\not\to$ su	7
3	qu $\not\to$ au	8
4	su $\not\to$ au	8

Table 14. Counter-Examples for Figure 14.
Y Compact and f Continuous

Item	Purpose	Example Number
1	au $\not\to$ su	7
2	au $\not\to$ k	11
3	qu $\not\to$ au	8
4	su $\not\to$ k	11
5	su $\not\to$ qu	16
6	c $\not\to$ qu	16
7	c $\not\to$ lu	16
8	k $\not\to$ su	19
9	lu $\not\to$ qu	9

Table 15. Counter-Examples for Figure 15.
X and Y Compact, f Continuous

Item	Purpose	Example Number
1	au $\not\to$ su	7
2	qu $\not\to$ au	8
3	su $\not\to$ au	8

Table 16. Counter-Examples for Figure 16.
f_a and f Continuous

Item	Purpose	Example Number
1	au $\not\to$ k	11
2	qu $\not\to$ au	22
3	su $\not\to$ qu	18
4	sc $\not\to$ qu	3
5	c $\not\to$ sc	4
6	sc $\not\to$ lu	17
7	k $\not\to$ pu	6
8	lu $\not\to$ sc	12
9	lu $\not\to$ qu	4

Table 17. Counter-Examples for Figure 17.
X Locally Compact, f_a and f Continuous

Item	Purpose	Example Number
1	au $\not\to$ k	11
2	sc $\not\to$ qu	3
3	qu $\not\to$ au	22
4	su $\not\to$ qu	18
5	c $\not\to$ sc	4

Table 18. Counter-Examples for Figure 18.
X Ccmpact, f_a and f Continuous

Item	Purpose	Example Number
1	au $\not\to$ k	11

Table 19. Counter-Examples for Figure 19.
Y Compact, f_a and f Continuous

Item	Purpose	Example Number
1	au $\not\to$ k	11
2	qu $\not\to$ au	22
3	lu $\not\to$ qu	9
4	c $\not\to$ lu	16

Table 20. Counter-Examples for Figure 20.
X and Y Compact, f_a and f Continuous

Item	Purpose	Example Number
1	au $\not\to$ k	11

Table 21. Counter-Examples for Figure 21.
$\{f_a\}$ Evenly Continuous

Item	Purpose	Example Number
1	au $\not\to$ sc	12
2	au $\not\to$ lu	10
3	qu $\not\to$ au	22
4	sc $\not\to$ lu	10
5	sc $\not\to$ qu	3
6	lu $\not\to$ qu	3
7	lu $\not\to$ sc	12

Table 22. Counter-Examples for Figure 22.
X Locally Compact, $\{f_a\}$ Evenly Continuous

Item	Purpose	Example Number
1	au $\not\to$ sc	11
2	sc $\not\to$ qu	3
3	qu $\not\to$ au	22

Table 23. Counter-Examples for Figure 24.
$\{f_a\}$ Uniformly Equicontinuous

Item	Purpose	Example Number
1	au $\not\to$ sc	21
2	au $\not\to$ lu	10
3	qu $\not\to$ au	22
4	lu $\not\to$ qu	3
5	lu $\not\to$ sc	21
6	sc $\not\to$ lu	10
7	sc $\not\to$ qu	3

CHAPTER III

RELATIONS BETWEEN VARIOUS TOPOLOGIES INDUCED BY CONVERGENCES

The convergences u, au, k, co, and p of Chapter I are topological in the sense that each type induces a topology on the family of functions F in which the convergence of nets in F coincides with the type of convergence.

John Brace [7] seems to be the only one to investigate the topology of almost uniform convergence. His work is in the space of all continuous functions from X into Y where Y is a locally convex linear topological space. We have placed the topology of almost uniform convergence in a more general setting and shown that almost uniform convergence is still topological in this setting.

The compact open topology seems to appear around 1945 in papers by Fox [12] and by Arens [1]. J. L. Kelley gives the credit for the beginning of the topology to Fox. Although in Fox [12], the compact open topology is referred to as a standard topology which one places on a function space.

It is our purpose in this chapter to investigate the relationships between the various topologies defined in the next section.

6. Definitions of the Topologies

Let A be a family of functions from X into $(Y, \mathcal{U})$. Let $(B{:}D) = \{f\colon\ f \in A \text{ and } f(B) \subset D\}$.

(6.1) Definition: The topology of pointwise convergence (P) has $\{(\{x\}{:}\ G)\colon x \in X \text{ and } G \text{ is open in } Y\}$ as a subbase.

(6.2) Definition: The compact open topology (CO) has $\{(K:G)$: K is compact in X and G is open in $Y\}$ as a subbase.

(6.3) Definition: For each V in $\mathcal{U}$ let $W_V = \{(f,g)$: f and g are in A and $(f(x), g(x))$ is in V for each x in $X\}$. The collection $\{W_V: V\epsilon\mathcal{U}\}$ is a base for the uniformity for the topology of uniform convergence on X (U).

(6.4) Definition: Let S be a family of subsets of X. For each V in $\mathcal{U}$ and B in S let $W_{V,B} = \{(f,g)$: $(f,g)\epsilon A\times A$ and $(f(x), g(x))$ is in V for each x in $B\}$. The collection $\{W_{V,B}: V\epsilon\mathcal{U}$ and $B\epsilon S\}$ is a subbase for the uniformity for the topology of uniform convergence on members of S (U(S)). When S is the set of all compact subsets of X then the resulting topology is the topology of uniform convergence on compacta (K). In this case the $W_{V,B}$'s form a base for the uniformity.

(6.5) Definition: Let B be a family of functions from X into $(Y,\mathcal{U})$. For each f in B and V in $\mathcal{U}$ define the collection $E(f,V,B)$ of subsets of B to be the set of all subsets U of B with the property that if $g_1,\ldots,g_n\epsilon B - U$ then there is an x in X such that $(g_k(x), f(x))$ is not in V for $k = 1,2,\ldots,n$. Let $N_f = \{E(f,V,B): V\epsilon\mathcal{U}\}$. The collection N_f is a subbase for the neighborhood system of f in the topology of almost uniform convergence (AUC) for the set B.

7. General Results

(7.1) Theorem: If $B \subset A \subset F(X,Y)$ then the relative topology for B in (A,AUC) is the same as the AUC topology on B.

Proof: Let T be the relative topology on B and let AUCB denote the topology of almost uniform convergence on B. Let U be in $E(f,V,A)$ and let $W = U \cap B$. W is T-open. If $g_1, g_2, \ldots, g_n \in B$ and not in W then $g_1, g_2, \ldots, g_n$ are not in U. Hence there is an x in X such that each $(g_i(x), g(x))$ is not in V. Thus W is in $E(f,V,B)$ and $T \subset$ AUCB.

Conversely, suppose that $W \subset B$ and W is in $E(f,V,B)$. The set W is open in AUCB. Let $U = W \cup (A-B)$. If $g_1, g_2, \ldots, g_n$ are in A and not in U then $g_1, g_2, \ldots, g_n$ are in B and not in W. Hence there is an x in X such that each $(g_i(x), f(x))$ is not in V. Therefore U is in $E(f,V,A)$. Since $W = U \cap B$ it follows that W is T-open and hence AUCB $\subset$ T.

(7.2) Definition: Let "s" denote a type of convergence. The convergence s is called topological for $A \subset F(X,Y)$ if and only if there is a topology T for A such that if $\{f_a\}$ is a net in A then f_a converges to f in the topology T if and only if f_a converges(s) to f.

(7.3) Theorem: Let X be a topological space, $(Y, \mathcal{U})$ be a uniform topological space and $A \subset F(X,Y)$. The convergences u, au, k, p and co are all topological for A.

Proof: The result is well known for u, k, p, and co. Brace [7, Theorem 2.4] has established the result for the au convergence when Y is a locally convex linear topological space. We will now show that the result holds in an arbitrary uniform space Y.

Suppose the net f_a converges to f in the AUC topology. Suppose that f_a doesn't converge (p) to f. Then there will be an x in X and a V in $\mathcal{U}$ such that $f_a(x)$ is frequently in $Y - V[f(x)]$. Hence there is a subnet $\{f_{a_b}\}$ in $Y - V[f(x)]$. Let $U = A - \{f_{a_b}\}$. Observe that f is in U which in turn is in $E(f,V,A)$, a contradiction. Thus the convergence must be pointwise. Suppose now that the convergence is pointwise but not quasi-uniform. There is a V in $\mathcal{U}$ and an index m such that for each $a_1, \ldots, a_n \geq m$ there is an x in X such that $(f_{a_i}(x), f(x))$ is not in V for $i = 1,2,\ldots,n$. Let $U = A - \{f_a : a \geq m\}$. Note that $f \in U \in E(f,V,B)$ thereby contradicting the convergence in the AUC topology.

Conversely suppose that f_a converges (au) to f. Suppose that f_a doesn't converge to f in the AUC topology. There is a base neighborhood $U = \bigcap_{i=1}^{n} U_i$ of f, where each $U_i \in E(f,V_i,A)$, such that f_a is frequently in $A - U$. Hence there is some integer i for which f_a is frequently in $A - U_i$. There exists a subnet $\{f_{a_b}\}$ in $A - U_i$. Let b' be an index. Since the subnet converges (qu) to f there are $b_1, b_2, \ldots, b_n \geq b'$ such that for each x in X there is an integer j so that $(f_{a_{b_j}}(x), f(x))$ is in V_i. But $f_{a_{b_j}}$ is in $A - U_i$ for $j = 1,2,\ldots,n$. Thus there is an s in X such that $(f_{a_{b_j}}(s), f(s))$ is not in V_i for $j = 1,2,\ldots,n$, a contradiction.

The next theorem is useful in comparing the topologies on a function space.

(7.4) <u>Theorem</u>: Let X and Y be topological spaces. Let A be a set of functions from X into Y. Let k and k' be topological convergences whose topologies are K and K' rsp. Then $K \subset K'$ if and only if the convergence k' implies the convergence k.

Immediate consequences of the convergence diagrams are the following theorems.

(7.5) Theorem: For a fixed set of functions A it follows that $P \subset AUC \subset U$, $P \subset CO$, and $P \subset K \subset U$. No other inclusion holds in general. The inclusions are strict.

Proof: See Figure 1 and Table 1.

(7.6) Theorem: If $A \subset C(X,Y)$ then $P \subset K = CO \subset U$ and $P \subset AUC \subset U$. No other inclusions hold in general. The inclusions are strict.

Proof: See Figure 16 and Table 16.

(7.7) Theorem: If $A \subset F(X,Y)$ and X is compact then $P \subset AUC \subset U = K$ and $P \subset CO$. No other inclusions hold. The inclusions are strict.

Proof: See Figure 3 and Table 3.

(7.8) Theorem: If $A \subset C(X,Y)$ and X is compact then $P = AUC \subset K = U = CO$.

Proof: See Figure 18 and Table 18.

Remark: Part of the preceding result is obtained in Brace [7, Theorem 4.2] when Y is a locally convex linear topological space.

(7.9) Theorem: If $A \subset C(X,Y)$, X is compact and A is an evenly continuous family (or equicontinuous) then $P = U = AUC = K = CO$ on A.

Proof: See Figure 23.

It would be interesting to know when are some of the other convergences topological on A. Immediately available from the convergence diagrams are the following results.

(7.10) Theorem: (i) If X is compact and $A \subset F(X,Y)$ then pu and lu are topological on A. (ii) If X is compact and $A \subset C(X,Y)$ then all of the eleven convergences are topological on A. (iii) If X is locally compact and $A \subset F(X,Y)$ then lu and pu are topological on A. (iv) If A is evenly continuous then c, pu, and su are topological on A. (v) If X is locally compact and $A \subset C(X,Y)$ then c, lu, su and pu are topological on A.

Remark: Suppose X is completely regular. Arens [1, Theorem 3] has shown that a necessary condition for the c convergence to be topological is that X be locally compact.

Arzela's Theorem (Theorem 2.1) concerns a net in $(C(X,Y),P)$ converging to a continuous limit. An inspection of the convergence diagrams led to the following analogy for a net in $(F(X,Y),K)$ to converge to a continuous limit.

(7.11) Theorem: Let X be locally compact and $(Y, \mathcal{U})$ be a uniform topological space. Let $\{f_a\}$ be a net in $F(X,Y)$ which converges to f. The function f is continuous if and only if the convergence is continuous.

Proof: If f is continuous then k convergence implies c con- by Figure 12. Lemma (4.1) guarantees that the continuity of f follows from c convergence.

Arens [1] and others have investigated "admissible" topologies for function spaces. The convergence diagrams give some insight into the problem of whether or not a topology is admissible.

(7.12) Definition: Let $A \subset F(X,Y)$. The topology T for A is admissible if and only if the convergence of a net $\{f_a\}$ in (A,T) to f implies that f_a converges (c) to f. In the terminology of Kelley [15], the topology T is jointly continuous.

(7.13) Theorem: For general topological spaces X, $(Y, \mathcal{U})$ and $A \subset F(X,Y)$ neither U, K, P, CO, nor AUC need be admissible topologies for A.

Proof: See Figure 1.

The following well known result is immediately available from Figure 1.

(7.14) Theorem: If $A \subset F(X,Y)$ and T is an admissible topology for A then $K \subset T$, and $CO \subset T$.

(7.15) Theorem: If X is locally compact and $(Y, \mathcal{U})$ is a uniform topological space and $A \subset C(X,Y)$ then the compact open topology is an admissible topology. The c convergence is topological and coincides with the compact open convergence (co), and the k convergence. The U topology is admissible but P and AUC need not be.

Proof: See Figure 17.

Remark: Parts of the preceding theorem are found in Arens [1, Theorem 2], Myers [17, Lemma 2.3] and Mrowka [16].

CHAPTER IV

PROPERTIES OF CERTAIN TOPOLOGIES

After having defined various topologies for a function space as in Chapter III one naturally wonders what kind of properties are possessed by such topologies. The question is difficult to answer even in very special circumstances. We obtain several partial results along this line.

8. The Pointwise Topology

Although the topology of pointwise convergence is the simplest to define, it has the fewest topological properties even under strong conditions on X, Y, and F. The following example gives some insight into the problem.

Example: Consider the space $(C([0,1],[0,1]),P)$. The following facts are known: (i) The space is separable. (ii) The space is not first countable. (iii) The space is Hausdorff and regular. (iv) The space is neither compact nor complete.

Let $A \subset F(X,Y)$. One of the tools used in the analysis of function spaces is the projection map P_x from A into Y defined by $P_x(f) = f(x)$ for each f in A and for each x in X.

(8.1) Theorem: Let X and Y be topological spaces, and let $A \subset F(X,Y)$. The projection map P_x from (A,P) into Y is a continuous map.

Proof: The space A can be considered as a subspace of the product space $\Pi\{Y_x$: x is in X$\}$ with the product topology where $Y_x = Y$ for each x in X. It is known (see Kelley [15, p.90]) that the projection P_x into the x^{th} coordinate space is a continuous and open map for each fixed x. Hence each P_x is continuous on the subspace A.

The preceding proof contains a general method for obtaining many properties of the topology of pointwise convergence from known properties of product spaces.

If $A = F(X,Y)$ we know that the projection map P_x is an open map onto Y. Unfortunately if A is a proper subset of $F(X,Y)$ then P_x need not be open even relative to its range, as is demonstrated in the next example.

Example: Let $X = \{1,2\}$ and $Y = [0,5]$. Define the sets A, A' and A'' by

$$A' = \{f: f \epsilon F(X,Y) \text{ and } 0 \leq f(1) = f(2) \leq 2\}$$

$$A'' = \{f: f \epsilon F(X,Y) \text{ and } 2 < f(1) \leq 3 \text{ and } f(2) = f(1) + 2\}$$

$$A = A' \cup A''.$$

The set A' is an open subset of A since the set U defined by

$U = \{f: f \epsilon A \text{ and } 0 \leq f(2) < 3\}$ is an open set in A and $A' = U \cap A$. Now $P_1(A) = [0,3]$ and $P_1(A') = [0,2]$ which is not an open set in $P_1(A)$ although the map P_1 is a one-to-one continuous function.

We have found a special condition which guarantees that P_x is open relative to its range. The condition also guarantees that P_x has certain other useful properties. The condition was motivated by the continuity in the initial conditions possessed by families of solutions to certain differential equations. Our condition is not as strong as equicontinuity on A. We state the condition along with a weaker condition to be used later.

(8.2) <u>Condition</u>: Let X be a topological space, $(Y, \mathcal{U})$ be a uniform topological space and let $A \subset F(X,Y)$. For each f in A there is an a in X such that for each finite subset $\{x_1, x_2, \ldots, x_n\}$ of X and for each V in $\mathcal{U}$ there is a U in $\mathcal{U}$ such that if g is in A and $(f(a), g(a))$ is in U then $(f(x), g(x))$ is in V for each $x = x_1, x_2, \ldots, x_n$.

(8.3) <u>Condition</u>: Let X be a topological space, $(Y, \mathcal{U})$ be a uniform topological space and let $A \subset F(X,Y)$. There is an a in X such that for each finite subset $\{x_1, x_2, \ldots, x_n\}$ of X, f in A and V in $\mathcal{U}$ there is a U in $\mathcal{U}$ such that if g is in A and $(f(a), g(a))$ is in U then $(f(x), g(x))$ is V for each $x = x_1, x_2, \ldots, x_n$.

<u>Remark</u>: If Y is T_1 then the preceding condition implies that P_a is one-to-one.

(8.4) <u>Theorem</u>: Under Condition (8.3) the map P_a is open onto its range.

<u>Proof</u>: Let f be in A and let $G = \{h\colon h \epsilon A$ and $(f(x_k), h(x_k))$ is in V for $k = 1, 2, \ldots, n\}$ where V is in $\mathcal{U}$ and each x_k is in X. Let U be as in Condition (8.3). Let $H = (U \cap V)[f(a)] \cap (P_a(A))$. If g is in $P_a^{-1}(H)$ then $P_a(g) = g(a) \epsilon H$. It follows that $(f(a), g(a))$ is in U and therefore $(f(x_k), g(x_k))$ is in V for each $k = 1, 2, \ldots, n$. Hence $g \epsilon G$ and $f \epsilon P_a^{-1}(H) \subset G$. Thus $P_a(f) \epsilon H \subset P_a(G)$.

(8.5) <u>Corollary</u>: If Y is T_1 then P_a is a homeomorphism under the conditions of Condition (8.3).

(8.6) <u>Theorem</u>: If A is the set of constant functions from X into

$(Y, \mathcal{U})$ then the projection map P_a is a homeomorphism from (A,P) onto Y.

Proof: Clearly P_a is one-to-one and continuous. We will show that P_a is open. Let f be in A, V be in $\mathcal{U}$ and $G = \{g\colon g \in A$ and $(f(x_k), g(x_k)) \in V$, for $K = 1,2,\ldots,n\}$ be a neighborhood of f. If y is in $V[f(a)]$ then let h be the constant function whose value is y. It follows that h is in G and $P_a(h) = h(a) = y$. Therefore $V[f(a)] \subset P_a G$.

(8.7) Corollary: If $A \subset F(X,Y)$, if A contains the constant functions, and if (A,P) is metrizable, compact, T_1, T_2, connected, first countable, second countable, completely regular, or separable metric then Y has the corresponding property.

Proof: The connected and compact properties follow from the continuity of P_x. The remaining properties are inherited by the subspace of constant functions in A and hence the homeomorphic image $P_x(A)$ inherits these properties.

(8.8) Theorem: Let $A \subset F(X,(Y,\mathcal{U}))$ and let A satisfy Condition (8.3). Then $P_a[A]$ is second countable if (A,P) is.

Proof: Let H be a countable base for Y. For each W in H let $B_W = \{f\colon f \in A$ and $f(a) \in W\}$. Let G be the set $\{g\colon g \in A$ and $g(x_k) \in W_k$ for $k = 1,\ldots,n\}$, where each W_k is in H and each x_k is in X. If f is in G then there are $V_1, V_2, \ldots, V_n$ in such that $V_k[f(x_k)] \subset W_k$ for $k = 1,2,\ldots,n$. Let $V = \bigcap_{k=1}^{n} V_k$ and let U be as in Condition (8.3). For some W in H we have $f(a) \in W \subset U[f(a)]$. Let g be in B_W. Hence $g(a) \in W$ so that $(f(a), g(a))$ is in U and hence each $(f(x_k), g(x_k))$ is in V for $k = 1,2,\ldots,n$. It follows that $g(x_k)$ is in $V_k[f(x_k)] \subset W_k$ for each

$k = 1,2,\ldots,n$. Thus $g \in G$ and $f \in B_W \subset G$. The desired countable base is $\{B_W : W \in N\}$.

(8.9) Corollary: A similar result follows when X is first countable and A satisfies Condition (8.2)

9. The Compact Open Topology

Much is known about the compact open topology on families of continuous functions. See for example Kelley [15], Dugundji [illegible], Warner [21], Arens [1], and Fox [12]. We obtain several results concerning the topology. Outstanding among these is Theorem ([illegible]) on [illegible]ability.

Let X be a topological space, $(Y, \mathcal{U})$ be a uniform topological space and let $A \subset F(X,Y)$.

(9.1) Theorem: The projection map P_x is a continuous map from (A, CO) into Y.

Proof: P_x is continuous from (A,P) into Y and $P \subset CO$.

(9.2) Theorem: If A is a set of constant functions from X into Y then P_x is a homeomorphism from (A,CO) onto $P_x(A)$.

Proof: Clearly P_x is one-to-one and continuous. Let $G = \{g : g \in A$ and $g(K_i) \subset U_i$ for $i = 1,2,\ldots,n\}$ where each K_i is compact and each U_i is open. Let $f \in G$. It follows that $f(t) \in U_i$ for each t in X and $i = 1,2,\ldots,n$. There is a V in $\mathcal{U}$ such that $V[f(x)] \subset \bigcap_{i=1}^{n} U_i$. If $y \in V[f(x)] \cap P_x(A)$ then let $g(t) = y$ for all t in X. The function g

is in G and thus y is in $P_x(G)$. Therefore $V[P_x(f)] \cap P_x(A) \subset P_x(G)$. Hence P_x is open and consequently a homeomorphism.

(9.3) <u>Theorem</u>: If $A \subset F(X,Y)$, if A contains the constant functions and if (A, CO) is metrizable, pseudo-metrizable T_1, T_2 compact, connected, first countable, second countable, regular, or completely regular then Y must have the corresponding property.

<u>Proof</u>: See the proof of Corollary (8.7).

A variant of the conditions in the previous section is useful in obtaining results in the CO topology.

(9.4) <u>Condition</u>: Let $A \subset F(X,Y)$. For each f in A there is an a in X such that for each compact set $K \subset X$ and for each V in $\mathcal{U}$ there is a U in $\mathcal{U}$ such that if g is in A and $(f(a), g(a))$ is in U then $(f(x), g(x))$ is in V for each x in K.

(9.5) <u>Condition</u>: Let $A \subset F(X,Y)$. There is an a in X such that for each compact $K \subset X$ and for each V in $\mathcal{U}$ there is a U in $\mathcal{U}$ such that if f and g are in A and $(f(a), g(a))$ is in U then $(f(x), g(x))$ is in V for each x in K.

(9.6) <u>Theorem</u>: Let A be a set of compactness preserving functions satisfying Condition (9.5). Then the projection map P_a is an open map from (A,CO) onto $P_a(A)$.

<u>Proof</u>: Let $W = \{h: h \in A \text{ and } h(K_i) \subset G_i \text{ for } i = 1,2,\ldots,n\}$ where each K_i is compact and each G_i is open. Let f be in W. Let $K = \bigcap_{i=1}^{n} K_i$. By Lemma (4.10) there is a V in $\mathcal{U}$ such that for $i = 1,2,\ldots,n$

if y is in $f(K_i)$ and (y,y') is in V then y' is in G_i. Let U and a be as in Condition (9.5). Let $N = U[f(a)] \cap P_a(A)$. If y is in N then $y = g(a)$ for some g in A and $(f(a), g(a))$ is in U. Hence $(f(x), g(x))$ is in V for each x in K_i and for each $i = 1,2,\ldots,n$. Thus $g(x)$ is in G_i for each x in K_i for $i = 1,2,\ldots,n$. Therefore g is in W. Thus y is in $P_a(W)$ so that $P_a(f)$ is in $N \subset P_a(W)$.

(9.7) Theorem: Let A be a set of compactness preserving functions satisfying Condition (9.4). If Y is first countable then so is (A,CO).

Proof: Let $g \epsilon A$. Let a be as in Condition 9.4. Let $\{U_k : k = 1,2,\ldots\}$ be a base for the neighborhood system at $g(a)$. Let $B_k = \{h: h \epsilon A \text{ and } h(a) \epsilon U_k\}$, for $k = 1,2,\ldots$ Let $W = \{f: f \epsilon A \text{ and } f(K_i) \subset G_i \text{ for } i = 1,2,\ldots,n\}$, where each K_i is compact and each G_i is open. Let g be in W. By Lemma (4.10) there is a V in $\mathcal{U}$ such that for $i = 1,2,\ldots,n$ if y is in $g(K_i)$ and (y,y') is in V then y' is in G_i. Let $K = \bigcup_{i=1}^{n} K_i$ and let U be as in Condition (9.4). There is a member of the neighborhood base U_r such that $g(a) \epsilon U_r \subset U[g(a)]$. Let $h \epsilon B_r$. It follows that $h(a)$ is in U_r and thus $(g(a), h(a))$ is in U so that $(g(x), h(x))$ is in G_i for each x in K_i. The function h is in W and therefore $g \epsilon B_r \subset W$.

(9.8) Theorem: Let A be a set of compactness preserving functions satisfying Condition (9.5). If Y is second countable then so is (A,CO).

A proof similar to the preceding one can be constructed.

Remark: If the topologies CO and K coincide it is obvious that both topologies are uniform topologies, as is the case when $A \subset C(X,Y)$. For spaces of noncontinuous functions it is not clear whether the CO topology is a uniform topology. Example (5.13) shows that even if A consists of compactness preserving functions that the CO and K topologies do not necessarily coincide. The following theorem gives a sufficient condition for (A, CO) to be a uniform space.

(9.9) Theorem: Let A be a subset of the compactness preserving functions on X into the uniform topological space $(Y, \mathcal{U})$. If A satisfies the Condition (9.4) then (A, CO) is a uniform topological space.

Proof: For each finite subset E of X and for each W in $\mathcal{U}$ let $M_{E,W} = \{f,g)$: f and $g \in A$, $(f(x), g(x)) \in W$ for each x in $E\}$. Let $B = \{M_{E,W}$: $E \subset A$, E is finite and W is a symmetric member of $\mathcal{U}\}$. We now show that B is a base for a uniformity for A. The following facts are readily verified: (i) The diagonal in $A \times A$ is a subset of each $M_{E,W}$; (ii) $M_{E \cup E', W \cap W'} \subset M_{E,W} \cap M_{E',W'}$; (iii) Given $M_{E,W}$ there is a W' so that $W' \circ W' \subset W$ and hence $M_{E,W'} \circ M_{E,W'} \subset M_{E,W}$; and (iv) $(M_{E,W})^{-1} = M_{E,W^{-1}}$. Hence B is a base for a uniformity S for A. Each $M_{E,W}[f]$ is open in the P topology and hence in the CO topology. Consequently $S \subset CO$. Now we will show the reverse inclusion. Let $H = \{g$: $g \in A$ and $g(K_i) \subset G_i$ for $i = 1,2,\ldots,n\}$ where each K_i is compact and each G_i is open. Let $K = \bigcup_{i=1}^{n} K_i$ and $f \in H$. By Lemma (4.10) there is a member V of $\mathcal{U}$ such that for $i = 1,2,\ldots,n$ if y is in $f(K_i)$ and (y,y') is in V then y' is in G_i. Let U and a be as in Condition (9.4). If g is in $M_{\{a\},U}[f] \cap A$ then $(f(a), g(a))$ is in U. If x is in K_i then $(f(x), g(x))$ is in V. Thus $g(x)$ is in G_i and therefore g is in H. It follows that f is in

$M_{\{a\},U}[f] \subset H$. Thus $CO \subset S$ and consequently, $(A,S) = (A,CO)$.

(9.10) <u>Theorem</u>: Let A be a set of compactness preserving functions from a topological space X into the pseudo-metric space (Y,d). If A satisfies Condition (9.5) with respect to the uniformity $\mathcal{U}$ generated by d then (A,CO) is pseudo-metrizable.

<u>Proof</u>: Let a be as in Condition (9.5). Let $M_n = \{(f,g): f$ and $g \in A$ and $d(f(a), g(a)) < 1/n\}$ for $n = 1,2,\ldots$ The family $\{M_n: n = 1,2,\ldots\}$ is a countable base for the uniformity giving the CO topology. The details can be patterned after the proof of the preceding theorem. The assertion will then be proven since the pseudo-metrizability of (A,CO) is equivalent to the uniformity for (A,CO) having a countable base (see Kelley [15, p. 186]).

In Warner [21, Theorem 5] necessary and sufficient conditions are given for the space of continuous real valued functions on X to be separable. His work depends heavily on the metric properties of the range. We have generalized the setting somewhat in the following.

(9.11) <u>Theorem</u>: If (X,d) is a compact metric space and Y is a separable locally convex linear topological space then $(C(X,Y), K)$ is separable.

<u>Proof</u>: A subset B of X will be called an ϵ-net of X if and only if for each x in X there is a b in B such that $d(x,b) < \epsilon$. Let Q be a countable dense subset of Y and let $\{P_n\}$ be a sequence of $(1/n)$-nets of X where $P_n = \{x_{n1}, x_{n2}, \ldots, x_{nr_n}\}$. For each subset $S = \{y_1, y_2, \ldots, y_{r_n}\}$ of Q define the function $J_n(S)$ from X into Y by

$$J_n(S)(x) = \frac{\sum_{i=1}^{r_n} m_{ni}(x)y_i}{\sum_{i=1}^{r_n} m_{ni}(x)}, \quad \text{where}$$

$m_{ni}(x) = 1/n - d(x,x_{ni})$ when $d(x,x_{ni}) \leq 1/n$ and

$m_{ni}(x) = 0$ otherwise,

for $i = 1,2,\ldots,r_n$ and $n = 1,2,3,\ldots$ We will now show that $\{J_n(S);\ n = 1,2,\ldots,\ S \subset Q$, and S has r_n elements$\}$ is a countable dense subset in $(C(X,Y),\ K)$. Let $f\epsilon C(X,Y)$ and let $H = \{g{:}g\epsilon C(X,Y)$ and $g(x) - f(x)\epsilon V$ for each x in $X\}$ where V is a neighborhood of the zero element θ. There is a convex neighborhood W of θ such that $W+W \subset V$. The function f is uniformly continuous on X hence there is a number $b > 0$ such that if $d(s,t) < b$ and $s,\ t\epsilon X$ then $f(s) - f(t)\epsilon W$. Choose an integer n such that $1/n < b$. For $i = 1,2,\ldots,r_n$ choose y_i in Q such that $y_i - f(x_{ni})$ is in W. Let $S = \{y_1,\ldots,y_{r_n}\}$. Let x be in X. If j is an integer such that $m_{nj}(x) \neq 0$ then $d(x,x_{nj}) < 1/n$ and $y_j - f(x) = y_j - f(x_{nj}) + f(x_{nj}) - f(x)$. Therefore $y_j - f(x)$ is in $W + W$. Thus for all j such that $m_{nj}(x) \neq 0$ we have y_j in $f(x) + (W + W)$, a convex set. But $J_n(S)(x)$ is a convex combination of the y_j's in $f(x) + (W + W)$ and hence $J_n(S)(x)$ is in $f(x) + (W + W)$. It follows that $J_n(S)(x) - f(x)\epsilon W + W \subset V$ and that $J_n(S)\epsilon H$. The collection $\{J_n(S){:}\ n = 1,2,\ldots,\ S \subset Q$, and S has r_n elements$\}$ is the required countable dense set in $C(X,Y)$.

<u>Remark</u>: The idea for the weighting functions m_{ni} came from Jane Cronin's

proof of the Schauder fixed point theorem (Cronin [9]).

Dugundji's extension of Tietze's theorem, stated next, allows an extension of Theorem (9.11).

(9.12) Theorem: Let X be a metric space and let A be a closed subset of X. Let L be a locally convex linear topological space and let f be a continuous function from A into L. There is an extension f^* of f from X into L such that $f^*(X)$ is a subset of the convex hull of $f(A)$.

For a proof see Dugundji [10].

(9.13) Definition: The topological space X is hemicompact if and only if X is the union of a countable number of compact subsets of X such that each compact set in X is contained in at least one member of the union.

(9.14) Theorem: If X is a metric space which is hemicompact and Y is a separable locally convex linear topological space, then $(C(X,Y), K)$ is separable.

Proof: Let $X = \bigcup_{n=1}^{\infty} K_n$ where each K_n is compact and satisfies the criteria of the hemicompactness condition for X. It follows from Theorem (9.11) that each $C(K_n,Y)$ is separable in the K topology. Let $\{f_{nj}: j = 1,2,\ldots\}$ be dense in $(C(K_n,Y), K)$ for $n = 1,2,\ldots$ By Theorem (9.12) there is a continuous extension g_{nj} of f_{nj} to all of X. We will now show that $\{g_{nj}: j,n = 1,2,\ldots\}$ is a countable dense subset of $(C(X,Y), K)$. Let K be a compact subset of X and let f be in $C(X,Y)$. Let

$H = \{g: g\varepsilon C(X,Y)$ and $g(x) - f(x)\varepsilon V$ for each x in $K\}$ where V is a neighborhood of θ. Choose an integer n such that $K \subset K_n$. Let $H^* = \{g: g\varepsilon C(K_n, Y)$ & $g(x) - f(x)\varepsilon V$ for each x in $K_n\}$. There is an f_{nj} in H^* so that $f_{nj}(x) - f(x)$ is in V for each x in K_n. The corresponding g_{nj} is in H and therefore $\{g_{nj}: j, n = 1,2,\ldots\}$ is a countable dense subset of $(C(X,Y), K)$.

Example: $(C(R^n, (B[0,1], T)), K)$ is separable where $B[0,1]$ denotes the space of all bounded functions on $[0,1]$ into the real numbers, and T is the sup norm topology.

(9.15) Corollary: If X is a hemicompact metric space and Y is a locally convex separable linear topological space then $(C(X,Y), P)$ is separable.

Proof: $P \subset K$.

(9.16) Theorem: If (X,d) is a metric space and Y is a T_1 locally convex linear topological space then for $(C(X,Y), K)$ to be first countable X must be hemicompact.

Proof: Suppose $(C(X,Y), K)$ is first countable. There is a neighborhood system $\{B_n: n = 1,2,3,\ldots\}$ of the θ function.

We now show that if K and K' are compact subsets of X and W and W' are neighborhoods of θ in Y such that $W' \neq Y$ then, using the notation in section 6, $(K:W) \subset (K': W')$ implies that $W \subset W'$ and $K' \subset K$. Let y be in W. The constant function h whose value is y is in $(K: W)$. Therefore y is in W'. Hence $W \subset W'$. Now suppose there is an x

in $K' - K$. Let z be in $Y - W'$. Define the continuous function g on the set $K \cup \{x\}$ by $g(t) = \theta$ for all t in K and $g(x) = z$. By Theorem (9.12) there is an extension f of g to all of X such that g is continuous. Now f is in $(K: W)$ but f is not in $(K': W')$. Hence $K' \subset K$.

For each positive integer i there is a compact subset K_i of X and a neighborhood W_i of θ such that $W_i \neq Y$ and such that $\overline{0}\epsilon(K_i: W_i) \subset B_i$ where $\overline{0}$ denotes the constant function whose value is θ. For every neighborhood $(K: W)$ of $\overline{0}$ with $W \neq Y$ there is an integer i so that $(K_i: W_i) \subset (K: W)$. In other words for every compact subset K of X there is an integer i such that $K \subset K_i$. Since point sets are compact it follows that for each x in X there is an integer i so that $(K_i: W_i) \subset (\{x\}: W)$ and hence $x \epsilon K_i$. Therefore $X = \bigcup_{n=1}^{\infty} K_n$.

<u>Remark</u>: The preceding proof is a modification of a proof for a similar result for $C(X,R)$ in Arens [1, Theorem 8].

(9.17) <u>Corollary</u>: Under the conditions of Theorem (9.16) if $(C(X,Y), K)$ is metrizable (pseudo-metrizable) then X is hemicompact.

(9.18) <u>Theorem</u>: If $(C(X,Y), K)$ is separable then Y must be separable.

<u>Proof</u>: The map P_x from $(C(X,Y), K)$ onto Y is continuous. Hence the image is separable.

(9.19) <u>Lemma</u>: Let X and each Y_a for $a \epsilon B$ be topological spaces. The space $(F(X, \Pi\{Y_a: a \epsilon B\}), K)$ is homeomorphic to $\Pi\{(F(X,Y_a), K): a \epsilon B\}$. If the topological space Y is homeomorphic to Z then $(F(X,Y), K)$ is homeomorphic to $\mathbb{F}(X,Z)$, K).

Proof: An exercise in definitions.

(9.20) Theorem: Let X be a completely regular topological space. Then $(C(X,R), K)$ is a separable metrizable space if and only if X is hemicompact and each compact subset of X is metrizable.

Proof: See Warner [21, Corollary to Theorem 6].

(9.21) Theorem: If X is a completely regular metric space and Y is a T_1 locally convex linear topological space then $(C(X,Y), K)$ is a separable metrizable space if and only if Y is a separable metrizable space and X is hemicompact.

Proof: Suppose $(C(X,Y), K)$ is a separable metric space. It follows from Theorem (7.6) that $K = CO$ in this case. By Theorems (9.3) and (9.18) Y is a separable metric space. The space is hemicompact by Corollary (9.17).

Suppose Y is a separable metric space and that X is hemicompact. The space $(C(X,[0,1]), K)$ is separable and metrizable by Theorem (9.20). Let $Y_n = [0,1]$ for $n = 1,2,\ldots$ From Lemma (9.19) $\Pi\{(C(X,Y_n), K): n = 1,2,\ldots\}$ is separable metrizable and homeomorphic to $(C(X,\Pi\{Y_n: n = 1,2,\ldots\})$, K). Now Y being separable metric is homeomorphic to a subspace Z of $\Pi\{Y_n: n = 1,2,\ldots\}$ (see Kelley [15, p. 125]). By Lemma (9.19) $(C(X,Y), K)$ is homeomorphic to $(C(X,Z), K)$ which is a subspace of $(C(X,\Pi\{Y_n: n = 1,2,\ldots\}), K)$. Therefore $(C(X,Y),K)$ is a separable metrizable space.

10. The Topology of Uniform Convergence

Let X be a topological space, $(Y, \mathcal{U})$ be a uniform topological space, S be a family of subsets of X which cover X, A be a subset of $F(X,Y)$, and let $U(S)$ denote the topology of uniform convergence on the members of S for the space A.

(10.1) Theorem: The projection map P_x is a continuous function from $(A,U(S))$ into Y.

Proof: $P \subset U(S)$.

(10.2) Theorem: If A is a set of constant functions from X into Y then P_x is a homeomorphism from $(A,U(S))$ onto $P_x(A)$, for each x in X.

Proof: The result follows from Theorem (8.6) and the fact that $(A,U(S)) = (A,P)$.

(10.3) Theorem: If $A \subset F(X,Y)$, if A contains the constant functions, and if $(A,U(S))$ is pseudo-metrizable, metrizable, compact, first countable, second countable, connected, T_1, or T_2 then Y has the corresponding property.

Proof: See the proof of Corollary (8.7).

(10.4) Theorem: Let S be a family of subsets of X for which there is a countable subfamily $\{D_i: i = 1,2,\ldots\}$ of S so that if D is in S then there is an integer i such that $D \subset D_i$. Then the space $(F(X,(Y,\mathcal{U})), U(S))$ is pseudo-metrizable (or metrizable) if and only if $(Y, \mathcal{U})$ is pseudo-metrizable (or metrizable).

Proof: The "only if" part follows from Theorem (10.3). Suppose

$(Y, \mathcal{V})$ is metrizable. Let d be a metric for Y for which $d(s,t) \leq 1$ for all s and t in Y. Define the metric b_i on $F(X,(Y,d))$ by $b_i(f,g) = \sup\{d(f(x),g(x)) : x \in D_i\}$ for $i = 1,2,\ldots,$ and for each f and g in $F(X,Y)$. Define the metric b by $b(f,g) = \sum_{i=1}^{\infty} 2^{-i} b_i(f,g)$. We now show that the metric topology for b is that of $U(S)$. Let $f \in F(X,Y)$ and let G be a neighborhood of f in the $U(S)$ topology of the form $\{h : h \in F(X,Y)$ and $d(f(x),h(x)) < r$ for each x in D_i for $i = 1,2,\ldots,n\}$. Notice that the collection of all such neighborhoods forms a base for $U(S)$. Observe that f is in $\{h : h \in F(X,Y)$ and $b(f,h) < r2^{-n}\} \subset G$. On the other hand let $M = \{h : h \in F(X,Y)$ and $b(f,g) < r\}$ be a neighborhood of f in the topology for the metric b on F. Choose a positive integer m so that $\sum_{i=m+1}^{\infty} 2^{-i} < r/2$, and let $q = r/2$. It follows that f is in $\{h : h \in F(X,Y)$ and $d(f(x),h(x)) < q$ for each x in D_i for $i = 1,2,\ldots,m\} \subset M$. Similarly for the pseudo metrizable case.

<u>Example</u>: The space $(C(R,R),U)$ is a non separable metric space. Although the conditions are not strong enough in this example to insure separability one can insure separability for the space of continuous functions which "vanish at ∞."

(10.5) <u>Definition</u>: Let X be a topological space and let Y be a linear topological space whose zero element is θ. A function f in $F(X,Y)$ <u>vanishes at ∞</u> if and only if for each neighborhood V of θ there is a compact subset K of X such that $f(X - K) \subset V$. Let $C_0(X,Y)$ denote the set of all continuous functions from X into Y which vanish at ∞.

(10.6) Theorem: Let Y be a locally convex linear topological space which is separable. The space $C_0(R^n, Y)$ is separable in the topology of uniform convergence.

Proof: The following proof was motivated by the proof of Theorem (9.11). Let $x = (x(1), \ldots, x(n)) \varepsilon R^n$, $\| x \| = (\sum_{i=1}^{n} x(i)^2)^{\frac{1}{2}}$, and let $d(x,y) = \| x - y \|$. For each $k = 1,2,\ldots$ let $\{x_{k1}, x_{k2}, \ldots\}$ be a countable $(1/k)$-net of R^n such that

(i) $i < j$ implies $\| x_{ki} \| \leq \| x_{kj} \|$, for $i,j,k = 1,2,\ldots$

(ii) Each bounded sphere contains at most a finite number of the x_{ki}'s, for any fixed k.

Let $i_{km} = \sup \{i : \| x_{ki} \| \leq m\}$ for each $m = 1,2,\ldots$ For $i, k = 1,2,\ldots$ define the functions m_{ki} by

$$m_{ki}(x) = 1/k - \| x - x_{ki} \| \text{ for } x \text{ in } X \text{ and } \| x - x_{ki} \| \leq 1/k$$

$$m_{ki}(x) = 0 \text{ for } x \text{ in } X \text{ and } \| x - x_{ki} \| > 1/k.$$

Let Q be a countable dense subset in Y. For each k, $m = 1,2\ldots$ and for each finite subset $T = \{y_1, y_2, \ldots, y_{i_{km}}\}$ of Q define the function $J(k,m,T)$ from X into Y by: For each x in R^n let

$$J(k,m,T)(x) = \frac{\sum_{i=1}^{\infty} m_{ki}(x) y_i}{\sum_{i=1}^{\infty} m_{ki}(x)}, \text{ where } y_i = \theta \text{ for } i > i_{km},$$

Note that for each x in R^n only finitely many of the $m_{ki}(x)$'s are non zero. We will show that the countable collection of all such $J(k,m,T)$'s is dense in $C_0(R^n, Y)$. Let H be the basic neighborhood of f defined by $H = \{h: h \in C_0(R^n, Y)$ and $(h(x) - f(x)) \in V$ for all x in $R^n\}$, where V is a symmetric neighborhood of θ in Y. There is a convex symmetric neighborhood W of θ such that $W + W + W \subset V$. Choose a positive integer m such that if $\|x\| > m$ then $f(x) \in W$. The function f is uniformly continuous on the set $M = \{x: x \in R^n$ and $\|x\| \leq m + 1\}$ hence there is a number $r > 0$ such that $r < 1/3$ and if s and t are in M with $\|s - t\| < r$ then $f(s) - f(t) \in W$. Choose a positive integer k such that $1/k < r$. For $i = 1, 2, \ldots, i_{km}$ choose $y_i \in Q$ so that $y_i - f(x_{ki}) \in W$. Let $T = \{y_1, \ldots, y_{i_{km}}\}$.

Let $\|x\| < m + 1/3$. If j is a positive integer such that $m_{kj}(x) \neq 0$ and $j \leq i_{km}$, then $\|x - x_{kj}\| < 1/k$ and $y_j - f(x) = (y_j - f(x_{kj})) + (f(x_{kj}) - f(x))$ and therefore $y_j - f(x) \in W + W$ (note: $x \in M$). Hence $y_j \in f(x) + (W + W)$, a convex neighborhood. If $j > i_{km}$ then $y_j = \theta$ and $y_j - f(x) = (\theta - f(x_{kj})) + (f(x_{kj}) - f(x)) \in W + W$ since $\|x\| > m$. Again $y_j \in f(x) + (W + W)$.

Let $\|x\| \geq m + 1/3$. It follows that $m_{kj}(x) = 0$ for all $j \leq i_{km}$ and thus for $j > i_{km}$ for which $m_{kj}(x) \neq 0$ we have $y_j = \theta$. Thus $y_j - f(x) = -f(x) \in W \subset W + W$ and therefore we have $y_j \in f(x) + (W + W)$, a convex neighborhood of $f(x)$. In all cases $J(k,m,T)(x)$ is a convex combination of the y_j's in the convex set $f(x) + (W + W)$ and therefore $J(k,m,T)(x)$ is also in $f(x) + (W + W)$ so that $J(k,m,T)(x) - f(x) \in V$ for all x in R^n. Consequently $J(k,m,T) \in H$.

<u>Remark</u>: See Corsin [8] for more results along this line.

The following conditions are analogous to Conditions (9.4) and (9.5).

(10.7) <u>Condition</u>: Let $A \subset F(X,Y)$ and S be a family of subsets of X covering X. For each f in A there is an a in X such that for each D in S and V in $\mathcal{V}$ there is a U in $\mathcal{V}$ such that if g is in A and $(f(a),g(a))$ is in U then $(f(x),g(x))$ is in V for each x in D.

(10.8) <u>Condition</u>: Let $A \subset F(X,Y)$. There is an a in X such that for each f in A, D in S and V in $\mathcal{V}$ there is a U in $\mathcal{V}$ such that if g is in A and $(f(a),g(a))$ is in U then $(f(x),g(x))$ is in V for each x in D.

(10.9) <u>Theorem</u>: If $A \subset F(X,Y)$ and A satisfies Condition (10.8) then P_a is an open map from $(A,U(S))$ onto $P_a(A)$. If Y is second countable then so is $(A,U(S))$.

<u>Proof</u>: Let $f \epsilon A$ and $W = \{h: h \epsilon A$ and $(f(x),h(x)) \epsilon V$ for each $x \epsilon D_i$ for $i = 1,2,\ldots,n\}$, be a neighborhood of f. There is a U in $\mathcal{V}$ such that $(f(a),g(a))$ is in U implies $(f(x),g(x))$ is in V for all x in D_i for $i = 1,2,\ldots,n$. Let $y \epsilon U[f(a)] \cap P_a(A)$. Then $y = g(a)$ for some g in A and $(f(a),g(a))$ is in U. Hence $(f(x),g(x))$ is in V for each x in D_i for $i = 1,2,\ldots,n$. Thus y is in $P_a(W)$ and $P_a(f)$ is in $U[f(a)] \cap P_a(A) \subset P_a(W)$.

Suppose $\{U_k: k = 1,2,\ldots\}$ is a countable base for Y. Let $B_k = \{g: g \epsilon A$ and $g(a) \epsilon U_k\}$. Let $f \epsilon A$ and $H = \{h: h \epsilon A$ and $(f(x),h(x))$ is in V for each $x \epsilon D_i$ for $i = 1,2,\ldots,n\}$ be a neighborhood of f. There is a U in $\mathcal{V}$ such that $(f(a),g(a))$ is in U implies $(f(x),g(x))$ is in V for all x in D_i for $i = 1,2,\ldots,n$. There is a U_k such that $f(a)$ is in $U_k \subset U[f(a)]$. It follows that $f \epsilon B_k \subset H$.

11. The Topology of Almost Uniform Convergence

Let X be a topological space, $(Y, \mathcal{U})$ be a uniform topological space and let A be a subset of $F(X,Y)$. Let (A, AUC) denote the set A with the topology of almost uniform convergence.

(11.1) Theorem: The projection map P_a is continuous from (A, AUC) into Y.

(11.2) Theorem: Let A be the subspace of constant functions in $F(X,Y)$ with the topology of almost uniform convergence, then $(A, AUC) = (A, P)$.

Proof: The convergence of a net pointwise in A implies uniform convergence of the net therefore $U \subset P$ and $P = AUC = U$.

(11.3) Corollary: Let A be the subspace of constant functions in $F(X,Y)$ with the "AUC" topology then P_a is a homemorphism from (A, AUC) onto $P_a(A)$.

(11.4) Corollary: If $A \subset F(X,Y)$, if A contains the constant functions and if (A, AUC) is metrizable, pseudo-metrizable, compact, connected, T_1, T_2, regular, completely regular, first countable, or second countable then Y has the corresponding property.

Proof: See the proof of Corollary (8.7).

(11.5) Corollary: If Y is T_1 or T_2 then so is (A, AUC).

Proof: Use the continuity of the map P_a.

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